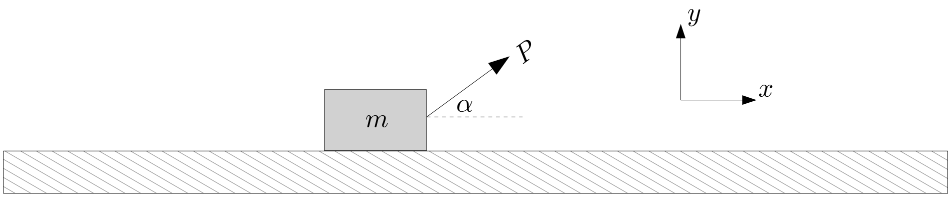
Box your answers.

1 [15pts] Pulling a Block

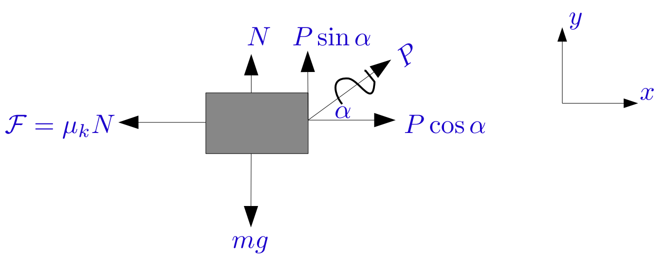


A block, of mass m, is pulled along a flat level plane, by a string that is attached to the block. The string pulls up at an angle α from the plane, with a constant force with a magnitude of P. The coefficient of kinetic fiction between the block and the plane is μ_k. The usual gravitational field acts down, $\vec{g} = -g\,\hat{y}$.

1.1 (5) Free Body Diagram

Draw a free body diagram of the block.

1.1 solution



1.2 (10) Acceleration

Find the acceleration, $\ddot{x}$, of the block.

1.2 solution

Applying Newton's second law we get

$$\sum F_x = P\cos\alpha - \mu_k N = m\ddot{x} \tag{1.1}$$

$$\sum F_y = N + P\sin\alpha - mg = 0 \tag{1.2}$$

where we assumed that the block is not moving in the y direction. Solving for $\ddot{x}$ by adding μ_k times equation 1.2 to equation 1.1 we get

$$P\cos\alpha + \mu_k P\sin\alpha - \mu_k mg = m\ddot{x} \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{P}{m}\left(\cos\alpha + \mu_k \sin\alpha\right) - \mu_k g}. \tag{1.3}$$

2 [10pts] Moment of Inertia

Find the moment-of-inertia tensor, **I**, for rotations about the origin for one particle, with mass m, that is located on the x axis at the position $(x, y, z) = (a, 0, 0)$. Write your answer in the form

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} . \tag{2.1}$$

2.0 solution

By inspection $I_{xx} = 0$, $I_{xy} = I_{yx} = I_{yz} = I_{zy} = I_{zx} = I_{xz} = 0$, and $I_{yy} = I_{zz} = m(x^2 + 0^2) = ma^2$. So

$$\boxed{\mathbf{I} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & ma^2 & 0 \\ 0 & 0 & ma^2 \end{bmatrix}} . \tag{2.2}$$

3 [20pts] Inverse Rocket (Again)

A large abandoned space ship travels through space which is filled with uniformly distributed "space dust," with mass density ρ. The only forces on the space ship are from the dust that collects on the ship as it goes through the dust. Consider the space dust to be at rest (not moving) before the ship hits it. Assume that all the dust that gets hit by the ship sticks to the ship and effectively increases the mass of the ship, and slows down the ship. All the motion is in one dimension.

3.1 (10) Find the Rate of Change of Mass of the Ship

A is the cross-sectional area of the ship that is passing (cutting) through the dust. Let m be the mass of the ship and the collected dust. Let v be the speed of the ship and the collected dust. Find rate of change of mass of the ship and the collected dust, $\frac{\mathrm{d}m}{\mathrm{d}t}$, as a function v, ρ, and A. Use the fact that the ship gains the mass of all the dust that it hits.

3.1 solution

As the ship moves at speed v

$$\mathrm{d}m = \rho\,\mathrm{d}V = \rho A v\,\mathrm{d}t \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A v} . \tag{3.1}$$

3.2 (10) Power

Find the rate at which kinetic energy is lost, P, as a function of v, ρ and A. In this case this power is the rate at which heat is generated on the front of the ship as the dust inelastically collides with it.

3.2 solution

Both the mass of the ship (plus collected dust), m, and the speed of the ship, v, are changing with time. So

$$P = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv^2\right) = -\frac{1}{2}v^2\dot{m} - vm\dot{v}\,. \tag{3.2}$$

From conservation of momentum (or the inverse rocket equation) we have

$$mv = \text{const} \quad \Rightarrow \quad m\dot{v} + v\dot{m} = 0\,. \tag{3.3}$$

With that and equation 3.2 we have

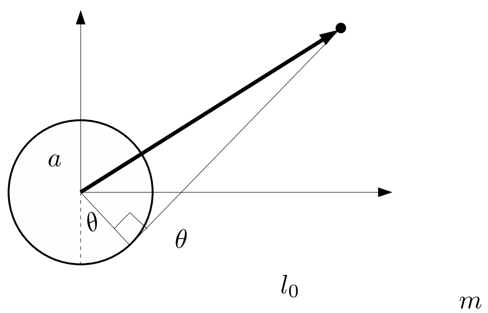
$$P = -\frac{1}{2}v^2\dot{m} - v\left(-v\dot{m}\right) = \frac{1}{2}v^2\dot{m} \tag{3.4}$$

and with equation 3.1 we get

$$P = \frac{1}{2}v^2\left(\rho A v\right) \quad \Rightarrow \quad \boxed{P = \frac{1}{2}\rho A v^3}\,. \tag{3.5}$$

This is the rate at which the kinetic energy of the system (ship plus dust) decreases.

4 [15pts] Tether-ball



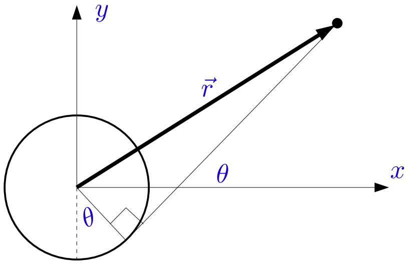
A tether-ball post has a radius a. A small ball with mass m swings on the end of the massless stretch-less rope. The length of the rope decreases as the angle θ increases, as the ball gets closer to the pole. Consider all the motion to be in the horizontal plane, so the effect of gravity is ignored. Think of it as a tether-ball on a smooth floor. The length of the rope when θ is zero is l_0.

4.1 (10) Lagrangian

Find the Lagrangian, $L(\theta, \dot{\theta})$, for the tether-ball system using θ as your generalized coordinate variable.

4.1 solution

$$L = T - U = T = \frac{1}{2}m\left(\dot{\vec{r}}\right)^2\,. \tag{4.1}$$



The length of the free rope will be $l_0 - a\theta$. If we choose the center of the pole to be the origin with x to the right and y up we get

$$\vec{r} = [a\sin\theta + (l_0 - a\theta)\cos\theta]\,\hat{x} + [-a\cos\theta + (l_0 - a\theta)\sin\theta]\,\hat{y}$$

$$\Rightarrow \quad \dot{\vec{r}} = \left[a\dot{\theta}\cos\theta - a\dot{\theta}\cos\theta - (l_0 - a\theta)\,\dot{\theta}\sin\theta\right]\hat{x} + \left[a\dot{\theta}\sin\theta - a\dot{\theta}\sin\theta + (l_0 - a\theta)\,\dot{\theta}\cos\theta\right]\hat{y}$$

$$= \left[-(l_0 - a\theta)\,\dot{\theta}\sin\theta\right]\hat{x} + \left[(l_0 - a\theta)\,\dot{\theta}\cos\theta\right]\hat{y} \Rightarrow \quad \left(\dot{\vec{r}}\right)^2 = (l_0 - a\theta)^2\,\dot{\theta}^2 = l_0^2\dot{\theta}^2 - 2l_0a\theta\dot{\theta}^2 + a^2\theta^2\dot{\theta}^2\,. \tag{4.2}$$

So

$$\boxed{L \quad = \quad \frac{1}{2}m\,(l_0 - a\theta)^2\,\dot{\theta}^2 \quad = \quad \frac{1}{2}ml_0^2\dot{\theta}^2 - ml_0a\theta\dot{\theta}^2 + \frac{1}{2}ma^2\theta^2\dot{\theta}^2}\,. \tag{4.3}$$

4.2 (5) Equation of Motion

Find the second order ordinary differential equation for the motion of θ.

4.2 solution

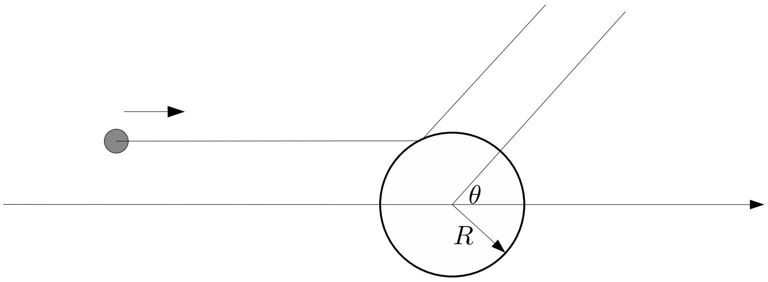
$$\frac{\partial L}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = 0 \quad \Rightarrow \quad -ml_0a\dot{\theta}^2 + ma^2\theta\dot{\theta}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(ml_0^2\dot{\theta} - 2ml_0a\theta\dot{\theta} + ma^2\theta^2\dot{\theta}\right) = 0$$

$$\Rightarrow \quad -ml_0a\dot{\theta}^2 + ma^2\theta\dot{\theta}^2 - \left(ml_0^2\ddot{\theta} - 2ml_0a\theta\ddot{\theta} - 2ml_0a\dot{\theta}^2 + 2ma^2\theta\dot{\theta}^2 + ma^2\theta^2\ddot{\theta}\right) = 0$$

$$\Rightarrow \quad ml_0a\dot{\theta}^2 - ma^2\theta\dot{\theta}^2 - ml_0^2\ddot{\theta} - ma^2\theta^2\ddot{\theta} + 2ml_0a\theta\ddot{\theta} = 0$$

$$\Rightarrow \quad \boxed{ml_0^2\ddot{\theta} + ma^2\theta^2\ddot{\theta} - 2ml_0a\theta\ddot{\theta} + ma^2\theta\dot{\theta}^2 - ml_0a\dot{\theta}^2 = 0}\,. \tag{4.4}$$

5 [20pts] Two Dimensional Hard Sphere Scattering



We will look at the analog of hard smooth sphere scattering, but with motion in just two dimensions. Consider all motion to be constrained to a plane, kind of like hockey pucks on an ice rink. We have smooth circular target particles with radius R. The target particles are fixed. The projectiles are small and scatter elastically off the target particles. There are $N_{\rm tar}$ sparsely distributed target particles over a length L along a direction that is perpendicular to a beam of small projectiles with a uniform intensity of $I_{\rm inc}$ (particles per unit length, per unit time). The width of the beam of particles is much larger than R, and smaller than all the target particles together (the whole target), as our scattering theory requires.

5.1 (10) Particles Not Scattered

Find the intensity (particles per unit length, per unit time) of the beam of small projectiles that go through the target undeflected, $I_{\rm un}$.

5.1 solution

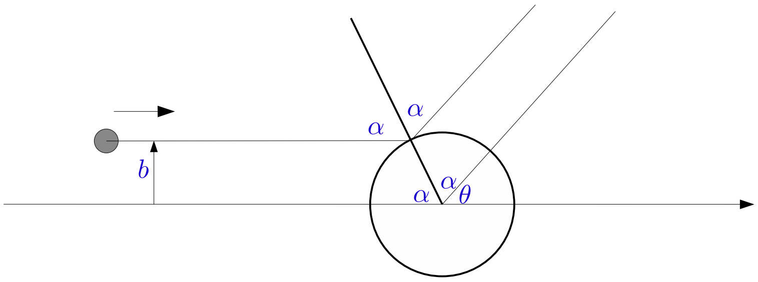
The total length of all targets is $N_{\rm tar}2R$. So the fraction of the total target length that is covered by targets is $\frac{2RN_{\rm tar}}{L}$. So

$$\boxed{I_{\rm un} = I_{\rm inc}\left(1 - \frac{2RN_{\rm tar}}{L}\right)}. \tag{5.1}$$

5.2 (10) Differential Cross Section

The width of the incident beam of particles is W ($W < L$ and $W \gg R$). Find the intensity (particles per unit length, per unit time) of the beam of small projectiles that is scattered at an angle θ at a distance D from the target ($D \gg L \gg R$), I_{sc}. In this case $-\pi \leq \theta \leq \pi$ and we don't need to consider solid angle. Hint: Start by finding the differential number of scatters, d$N_{\rm sc}$, as a function of the differential impact parameter, db, and then consider the impact parameter, b, to be a function of θ.

5.2 solution



Let $N_{\rm inc}$ be the number of projectiles particles incident on the target over some time Δt. Let the rate of incident particles be constant over time. So the incident intensity is $I_{\rm inc} = \frac{1}{\Delta t}\frac{{\rm d}N_{\rm inc}}{{\rm d}x}$, where x is the distance across the beam. Let $N_{\rm sc}$ be the number of particles that are scattered from the $N_{\rm inc}$ incident projectiles

$$ {\rm d}N_{\rm sc} = N_{\rm inc}\frac{2RN_{\rm tar}}{L}\frac{{\rm d}b}{2R} = N_{\rm inc}\frac{N_{\rm tar}}{L}\,{\rm d}b\,, \tag{5.2}$$

and from the above figure

$$ b = R\sin\alpha \qquad \text{and} \qquad 2\alpha+\theta=\pi \quad\Rightarrow\quad \alpha = \frac{\pi}{2}-\frac{\theta}{2}$$

$$ \Rightarrow \quad b = R\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = R\cos\frac{\theta}{2} \quad\Rightarrow\quad {\rm d}b = -\frac{1}{2}R\left(\sin\frac{\theta}{2}\right){\rm d}\theta\,.$$

So equation 5.2 may be rewritten as

$$ {\rm d}N_{\rm sc} = N_{\rm inc}\frac{N_{\rm tar}}{L}\,|{\rm d}b| = N_{\rm inc}\frac{N_{\rm tar}}{L}\frac{1}{2}R\left(\sin\frac{\theta}{2}\right){\rm d}\theta = \frac{1}{2}N_{\rm inc}\frac{RN_{\rm tar}}{L}\left(\sin\frac{\theta}{2}\right){\rm d}\theta\,. \tag{5.3}$$

We are measuring the intensity at a distance D from the target, so we will measure ${\rm d}\theta$ as $\frac{{\rm d}s}{D}$, where ${\rm d}s$ is a infinitesimal distance along a circular arc of radius D. So

$$ {\rm d}N_{\rm sc} = \frac{1}{2}N_{\rm inc}\frac{RN_{\rm tar}}{L}\left(\sin\frac{\theta}{2}\right)\frac{{\rm d}s}{D}\,. \tag{5.4}$$

So the intensity of scattered particles at angle θ is

$$ I_{sc} = \frac{1}{\Delta t}\frac{{\rm d}N_{\rm sc}}{{\rm d}s} = \frac{1}{2}\frac{N_{\rm inc}}{\Delta t}\frac{RN_{\rm tar}}{DL}\left(\sin\frac{\theta}{2}\right) \tag{5.5}$$

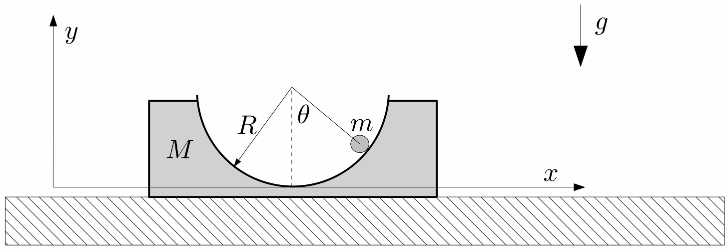
The time rate of incident particles make be replaced with a function of the uniform incident intensity, $I_{\rm inc}$ because

$$ \frac{N_{\rm inc}}{\Delta t} = \int_{x=0}^{W} I_{\rm inc}\,{\rm d}x = I_{\rm inc}W\,. \tag{5.6}$$

So

$$ I_{sc} = \frac{1}{2}\left(I_{\rm inc}W\right)\frac{RN_{\rm tar}}{DL}\left(\sin\frac{\theta}{2}\right) \quad\Rightarrow\quad \boxed{I_{sc} = \frac{I_{\rm inc}WRN_{\rm tar}}{2DL}\sin\frac{\theta}{2}}\,. \tag{5.7}$$

6 [20pts] Circular Slide



The figure above shows a particle with mass m that slides on a frictionless circular slide. The circular slide has a radius R. The circular slide is cut from a block of that slides without friction on a flat table. The mass of the circular slide block is M. There is a uniform gravitational field with magnitude g.

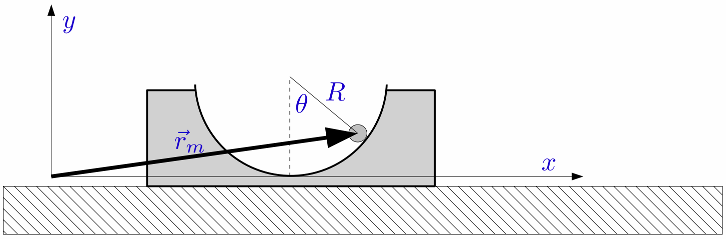
6.1 (10) Lagrangian

We define the x position of the center-of-mass, $x_{\rm cm}$, as

$$x_{\rm cm} \equiv \frac{m\,x_m + M\,x_M}{m+M} \tag{6.1}$$

where x_m is the x position of the particle and x_M is the x position of the center-of-mass of the block. Find the Lagrangian of this system using the generalized coordinate variables $x_{\rm cm}$ and θ, $L(\dot{x}_{\rm cm}, \theta, \dot{\theta})$. Hint: Start by writing T as a function of the speeds of the two objects separately, keeping in mind that the particle moves in circular motion relative to the block.

6.1 solution



The position of the particle, $\vec{r}_m$, is

$$\vec{r}_m = (x_M + R\sin\theta)\,\hat{x} + (R - R\cos\theta)\,\hat{y} \quad \Rightarrow \quad \dot{\vec{r}}_m = \left(\dot{x}_M + R\dot{\theta}\cos\theta\right)\hat{x} + \left(R\dot{\theta}\sin\theta\right)\hat{y} \tag{6.2}$$

$$\Rightarrow \quad \left(\dot{\vec{r}}_m\right)^2 = \dot{x}_M^2 + R^2\dot{\theta}^2 + 2R\dot{x}_M\dot{\theta}\cos\theta\,. \tag{6.3}$$

$$L = T - U = T_M + T_m - U = \frac{1}{2}m\left(\dot{\vec{r}}_m\right)^2 + \frac{1}{2}M\dot{x}_M^2 - (-mgR\cos\theta)$$

$$= \frac{1}{2}m\dot{x}_M^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{x}_M\dot{\theta}\cos\theta + \frac{1}{2}M\dot{x}_M^2 + mgR\cos\theta\,. \tag{6.4}$$

Note that with the Lagrangian written in terms of variables x_M, $\dot{x}_M$, θ, and $\dot{\theta}$ we will have coupled equations of motion for x_M and θ. We change variables to $x_{\rm cm}$, $\dot{x}_{\rm cm}$, θ, and $\dot{\theta}$: From equation 6.1 and 6.2 we can solve for $\dot{x}_M$ in terms of $\dot{x}_{\rm cm}$, θ, and $\dot{\theta}$ like so

$$(m+M)\,x_{\rm cm} = mx_m + Mx_M = m\,(x_M + R\sin\theta) + Mx_M = (m+M)\,x_M + mR\sin\theta$$

$$\Rightarrow \quad (m+M)\,x_M = (m+M)\,x_{\rm cm} - mR\sin\theta \quad \Rightarrow \quad x_M = x_{\rm cm} - \frac{m}{m+M}R\sin\theta$$

$$\Rightarrow \quad \dot{x}_M = \dot{x}_{\rm cm} - \frac{m}{m+M}R\dot{\theta}\cos\theta\,. \tag{6.5}$$

So

$$L = \frac{1}{2}(m+M)\left(\dot{x}_{\rm cm} - \frac{m}{m+M}R\dot{\theta}\cos\theta\right)^2 + \frac{1}{2}mR^2\dot{\theta}^2 + mR\left(\dot{x}_{\rm cm} - \frac{m}{m+M}R\dot{\theta}\cos\theta\right)\dot{\theta}\cos\theta + mgR\cos\theta$$

$$= \frac{1}{2}(m+M)\,\dot{x}_{\rm cm}^2 + \frac{1}{2}\frac{m^2}{m+M}R^2\dot{\theta}^2\cos^2\theta - mR\dot{x}_{\rm cm}\dot{\theta}\cos\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mR\dot{x}_{\rm cm}\dot{\theta}\cos\theta - \frac{m^2}{m+M}R^2\dot{\theta}^2\cos^2\theta + mgR\cos\theta$$

$$\Rightarrow \quad \boxed{L = \frac{1}{2}(m+M)\,\dot{x}_{\rm cm}^2 - \frac{1}{2}\frac{m^2}{m+M}R^2\dot{\theta}^2\cos^2\theta + \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta}\,. \tag{6.6}$$

So we see that L is separable, and the $x_{\rm cm}$ and θ motions are independent.

6.2 (5) Differential Equation of Motion

From your Lagrangian, find the differential equations of motion for $x_{\rm cm}$ and θ.

6.2 solution

For $x_{\rm cm}$

$$\frac{\partial L}{\partial x_{\rm cm}} - \frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{x}_{\rm cm}} = 0 \quad \Rightarrow \quad \frac{\rm d}{{\rm d}t}\left[(m+M)\,\dot{x}_{\rm cm}\right] = 0 \quad \Rightarrow \quad \boxed{\ddot{x}_{\rm cm} = 0}\,. \tag{6.7}$$

So the center-of-mass motion in the x direction is as expected.

$$\frac{\partial L}{\partial \theta} - \frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad \frac{m^2}{m+M}R^2\dot{\theta}^2\cos\theta\,\sin\theta - mgR\sin\theta - \frac{\rm d}{{\rm d}t}\left(-\frac{m^2}{m+M}R^2\dot{\theta}\cos^2\theta + mR^2\dot{\theta}\right) = 0$$

$$\Rightarrow \quad \frac{m^2}{m+M}R^2\dot{\theta}^2\cos\theta\,\sin\theta - mgR\sin\theta - \left(-\frac{m^2}{m+M}R^2\ddot{\theta}\cos^2\theta + 2\frac{m^2}{m+M}R^2\dot{\theta}^2\cos\theta\,\sin\theta + mR^2\ddot{\theta}\right) = 0$$

$$\Rightarrow \quad \boxed{mR^2\ddot{\theta} - \frac{m^2}{m+M}R^2\ddot{\theta}\cos^2\theta + \frac{m^2}{m+M}R^2\dot{\theta}^2\cos\theta\,\sin\theta + mgR\sin\theta = 0}\,. \tag{6.8}$$

6.3 (5) Find the Angular Frequency of Small Oscillations

Find the angular frequency for small θ oscillations, ω_0. Hint: $\theta\dot{\theta}^2 \approx 0$.

6.3 solution

For small oscillations

$$\sin\theta \approx \theta\,, \qquad \cos\theta \approx 1\,, \qquad \text{and} \qquad \theta\dot{\theta}^2 \approx 0 \tag{6.9}$$

where we get $\theta\dot{\theta}^2 \approx 0$ because $\dot{\theta}$ is of size of order θ. Using this the equation of motion for θ (equation 6.8) becomes

$$mR^2\ddot{\theta} - \frac{m^2}{m+M}R^2\ddot{\theta} + mgR\theta = 0 \quad \Rightarrow \quad \left(1 - \frac{m}{m+M}\right)\ddot{\theta} + \frac{g}{R}\theta = 0 \quad \Rightarrow \quad \ddot{\theta} = -\frac{m+M}{M}\frac{g}{R}\,\theta\,, \tag{6.10}$$

which is the differential equation for simple harmonic motion with an angular frequency of

$$\boxed{\omega_0 = \sqrt{\frac{m+M}{M}\,\frac{g}{R}}}\,. \tag{6.11}$$

Check limiting case, if M gets large, ω_0 is like the angular frequency of a simple pendulum (with small oscillations) with length R. When M is small ω_0 gets large, which seems to make sense.

In PHYS 3356 you'll learn an easier way to do this.