1 [20pts] Gravitation

A uniform solid sphere of mass M and a radius R is fixed a distance h from it's center above a thin infinitely long cylindrical mass with uniform mass density λ (mass/length). h is greater than R. What is the force on the sphere from the cylindrical mass? Using Gauss's law is the easiest way to do this problem, but it is not required. Anyone doing this the hard way may need

$$\int \frac{\mathrm{d}x}{\sqrt{(x^2 \pm a^2)^3}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}} + C \qquad \text{or} \qquad \int \frac{\mathrm{d}x}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \tag{1.1}$$

1.0 solution

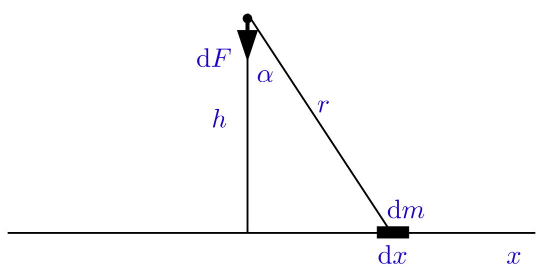
From symmetry the gravitational field from the infinitely long cylindrical mass will be radially inward from the cylindrical mass. We apply Gauss's law to the infinitely long cylindrical mass with a cylindrical Gaussian surface, with length l, around the long cylindrical mass with the long cylindrical mass is it's center. This gives

$$\oint \vec{g} \cdot \mathrm{d}\vec{A} = -4\pi G M_{\text{inside}} \quad \Rightarrow \quad g\,(2\pi r l) = -4\pi G\,(l\lambda) \quad \Rightarrow \quad g = -\frac{2G\lambda}{r} \tag{1.2}$$

The uniform solid sphere will act like a point mass of mass M at $r = h$. So the force on the on the sphere will be

$$M\vec{g} = \boxed{\frac{2GM\lambda}{h}\,\text{(toward the long cylindrical mass)}}. \tag{1.3}$$

Now the hard way. From symmetry the net force on the sphere will be in perpendicular to the long cylindrical mass.



$$F = \int \mathrm{d}F = 2\int_{x=0}^{\infty} G\frac{M\,\mathrm{d}m}{r^2}\cos\alpha = 2\int_{x=0}^{\infty} G\frac{M\,(\lambda \mathrm{d}x)}{x^2+h^2}\left(\frac{h}{\sqrt{x^2+h^2}}\right) = 2GM\lambda h\int_{x=0}^{\infty}\frac{\mathrm{d}x}{\sqrt{(x^2+h^2)^3}}$$

$$= 2GM\lambda h\int_{x=0}^{\infty}\frac{\mathrm{d}x}{\sqrt{(x^2+h^2)^3}} = 2GM\lambda h\left(\frac{x}{h^2\sqrt{x^2+h^2}}\right)\Bigg|_{x=0}^{\infty}$$

$$= \frac{2GM\lambda}{h}\left[\left(\frac{1}{\sqrt{1+\frac{h^2}{x^2}}}\right)\Bigg|^{x=\infty} - \left(\frac{x}{h\sqrt{1+\frac{x^2}{h^2}}}\right)\Bigg|^{x=0}\right] = \frac{2GM\lambda}{h}\,[1-0]$$

$$= \boxed{\frac{2GM\lambda}{h}\,\text{(toward the long cylindrical mass)}}. \tag{1.4}$$

2 [20pts] Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \left(y'^2 - y^2 \right) \mathrm{d}x, \tag{2.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. You may need

$$\int \frac{\mathrm{d}u}{\sqrt{a^2 - u^2}} = \cos^{-1} \frac{u}{|a|} + C \,, \quad a^2 > u^2 \,. \tag{2.2}$$

2.0 solution

We define

$$f = y'^2 - y^2 \,. \tag{2.3}$$

For J to be stationary

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial f}{\partial y'} = 0 \quad \Rightarrow \quad -2y - \frac{\mathrm{d}}{\mathrm{d}x} (2y') = 0 \quad \Rightarrow \quad y = -y'' \quad \Rightarrow \quad y = -\frac{\mathrm{d}y'}{\mathrm{d}y} \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\Rightarrow \quad y = -\frac{\mathrm{d}y'}{\mathrm{d}y} y' \quad \Rightarrow \quad y \, \mathrm{d}y = -y' \frac{\mathrm{d}y'}{\mathrm{d}y} \mathrm{d}y \quad \Rightarrow \quad \int y \, \mathrm{d}y = -\int y' \, \mathrm{d}y'$$

$$\Rightarrow \quad \frac{1}{2} y^2 = -\frac{1}{2} y'^2 + c_1^2 \quad \Rightarrow \quad y^2 - c_1^2 = -y'^2 \,, \tag{2.4}$$

where c_1^2 is a constant of integration that is greater than zero.

$$\Rightarrow \quad \sqrt{c_1^2 - y^2} = y' \quad \Rightarrow \quad \mathrm{d}x = \frac{y' \mathrm{d}x}{\sqrt{c_1^2 - y^2}} \quad \Rightarrow \quad \int \mathrm{d}x = \int \frac{\mathrm{d}y}{\sqrt{c_1^2 - y^2}} \,.$$

$$\Rightarrow \quad x = \cos^{-1} \frac{y}{c_1} + c_2 \quad \Rightarrow \quad \boxed{y = c_1 \cos (x - c_2)} \quad \text{or} \quad \boxed{y = c_1 \sin (x - c_2)} \,. \tag{2.5}$$

3 [30pts] Tension in a Simple Pendulum

We have a simple pendulum with length l, bob mass m, and in a uniform gravitational field g. Use Lagrangian dynamics with an undetermined multiplier, λ, and Lagrangian $L\left(r, \theta, \dot{r}, \dot{\theta}\right)$, where θ is the angle from the vertical, and r is the length of the pendulum, to find: **(a)** an equation of constraint, f, **(b)** the Lagrangian, **(c)** the equations of motion for θ and r, **(d)** λ, and **(e)** the tension, T, in the pendulum. All may be a function of r, $\dot{r}$, θ, $\dot{\theta}$, m, g, and l.

3.0 solution

The constraint equation is

$$(\mathbf{a}) \; \boxed{f = r - l = 0} \,. \tag{3.1}$$

$$L = T - U = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - mg \left(-l \cos \theta \right) \quad \Rightarrow \quad (\mathbf{b}) \; \boxed{L\left(r, \theta, \dot{r}, \dot{\theta}\right) = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + mgr \cos \theta} \,. \tag{3.2}$$

For the θ equation of motion we have

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} + \lambda \frac{\partial f}{\partial \theta} = 0 \quad \Rightarrow \quad mgl\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) + 0 = 0 \quad \Rightarrow \quad mr^2\ddot{\theta} = -mgr\sin\theta - 2mr\dot{r}\dot{\theta}\,. \tag{3.3}$$

With the constraint equation $r = l$ and $\dot{r} = 0$ so

$$ml^2\ddot{\theta} = -mgl\sin\theta - 0 \quad \Rightarrow \quad (\mathbf{c})\, \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}\,. \tag{3.4}$$

For the r equation of motion we have

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} + \lambda \frac{\partial f}{\partial r} = 0 \quad \Rightarrow \quad mr\dot{\theta}^2 + mg\cos\theta - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) + \lambda\,[1] = 0 \quad \Rightarrow \quad m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta + \lambda \tag{3.5}$$

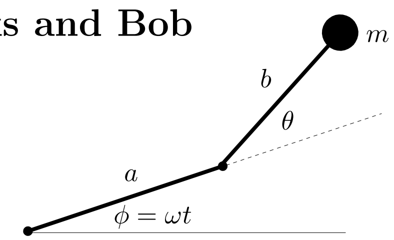
With the constraint equation $(\mathbf{c})\,\boxed{r = l}$, $\dot{r} = 0$, and $\ddot{r} = 0$ so

$$0 = ml\dot{\theta}^2 + mg\cos\theta + \lambda \quad \Rightarrow \quad (\mathbf{d})\, \boxed{-\lambda = ml\dot{\theta}^2 + mg\cos\theta} \tag{3.6}$$

The generalized force of constraint for r is $\lambda\frac{\partial f}{\partial r} = \lambda$ is a regular force in the r direction and since tension is in the $-r$ direction we have $T = -\lambda$. So with this and equation 3.6 we get

$$(\mathbf{e})\, \boxed{T = -\lambda = ml\dot{\theta}^2 + mg\cos\theta}\,. \tag{3.7}$$

4 [30pts] Swinging Sticks and Bob

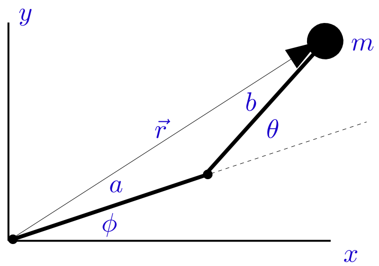


The two massless sticks are connected together at one end and they of free to pivot in a plane about that end. The other end of one of the sticks has a bob of mass m fixed to it. The other end of the other stick is fixed at a point, which it rotates about at a constant angular speed of ω. The stick without the bob on an end has a length a and the stick with the bob on the end has a length of b ($a \neq b$). All of the motion is in a plane. There is no gravity acting on the bob. Use θ as the generalized coordinate variable, as shown in the figure above. (Pretty much what you see in the figure above.)

4.1 (15pts) Lagrangian

Find the Lagrangian, $L(\theta, \dot{\theta})$ for this system. Note that $U = 0$.

4.1 solution



From the position, $\vec{r}$, of the bob we will get the kinetic energy of the bob. From the figure above (doing it the long way this time)

$$\vec{r} = [a\cos\phi + b\cos(\phi+\theta)]\,\hat{x} + [a\sin\phi + b\sin(\phi+\theta)]\,\hat{y}$$

$$\Rightarrow \quad \dot{\vec{r}} = \left[-a\omega\sin\phi - b\left(\omega+\dot{\theta}\right)\sin(\phi+\theta)\right]\hat{x} + \left[a\omega\cos\phi + b\left(\omega+\dot{\theta}\right)\cos(\phi+\theta)\right]\hat{y}$$

$$\begin{aligned}\Rightarrow \quad \left(\dot{\vec{r}}\right)^2 \quad = \quad & \Big[a^2\omega^2\sin^2\phi + b^2\left(\omega+\dot{\theta}\right)^2\sin^2(\phi+\theta) + 2ab\,\omega\left(\omega+\dot{\theta}\right)\sin\phi\,\sin(\phi+\theta) \\ & + a^2\omega^2\cos^2\phi + b^2\left(\omega+\dot{\theta}\right)^2\cos^2(\phi+\theta) + 2ab\,\omega\left(\omega+\dot{\theta}\right)\cos\phi\,\cos(\phi+\theta)\Big]\end{aligned}$$

$$\begin{aligned} = \quad & a^2\omega^2\left(\sin^2\phi+\cos^2\phi\right) + b^2\left(\omega+\dot{\theta}\right)^2\left[\sin^2(\phi+\theta)+\cos^2(\phi+\theta)\right] \\ & + 2ab\,\omega\left(\omega+\dot{\theta}\right)\left[\sin\phi\,\sin(\phi+\theta)+\cos\phi\,\cos(\phi+\theta)\right]\end{aligned}$$

$$= a^2\omega^2 + b^2\left(\omega+\dot{\theta}\right)^2 + 2ab\,\omega\left(\omega+\dot{\theta}\right)\cos\theta \tag{4.1}$$

So

$$L = T - U = \frac{1}{2}m\left(\dot{\vec{r}}\right)^2 - 0 = \frac{1}{2}ma^2\omega^2 + \frac{1}{2}mb^2\left(\omega+\dot{\theta}\right)^2 + mab\,\omega\left(\omega+\dot{\theta}\right)\cos\theta$$

$$\Rightarrow \quad \boxed{L(\theta,\dot{\theta}) = \frac{1}{2}ma^2\omega^2 + \frac{1}{2}mb^2\left(\omega+\dot{\theta}\right)^2 + mab\,\omega\left(\omega+\dot{\theta}\right)\cos\theta}. \tag{4.2}$$

4.2 (10pts) θ Equation of Motion

Find the equation of motion for θ ($\ddot{\theta} = ?$).

4.2 solution

$$\frac{\partial L}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{\theta}} = 0 \quad \Rightarrow \quad -mab\,\omega\left(\omega+\dot{\theta}\right)\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left[mb^2\left(\omega+\dot{\theta}\right) + mab\,\omega\cos\theta\right] = 0$$

$$\Rightarrow \quad -mab\,\omega\left(\omega+\dot{\theta}\right)\sin\theta - \left(mb^2\ddot{\theta} - mab\,\omega\dot{\theta}\sin\theta\right) = 0$$

Dividing by mab and rearranging gives

$$\Rightarrow \quad \frac{b}{a}\ddot{\theta} = -\omega\left(\omega+\dot{\theta}\right)\sin\theta + \omega\dot{\theta}\sin\theta \quad \Rightarrow \quad \boxed{\ddot{\theta} = -\frac{a}{b}\,\omega^2\sin\theta}. \tag{4.3}$$

4.3 (5pts) Equilibrium and Stability

Find the equilibrium θ positions θ_0. Determine if these positions are stable or not. For any stable positions, find the angular frequency of oscillation about that equilibrium position.

4.3 solution

$$\ddot{\theta}\,(\theta_0) = 0 \quad \Rightarrow \quad \sin\theta_0 = 0 \quad \Rightarrow \quad \boxed{\theta_0 = 0\,,\pi}\,. \tag{4.4}$$

Checking stability of $\theta_0 = 0$

$$\left.\frac{\mathrm{d}\ddot{\theta}}{\mathrm{d}\theta}\right|_{\theta=0} = -\frac{a}{b}\omega^2 \sin 0 = -\frac{a}{b}\omega^2\,, \tag{4.5}$$

so $\theta_0 = 0$ is stable and our system has an angular frequency of small oscillations about $\theta = 0$ of $\omega_0 = \sqrt{\frac{a}{b}\,\omega^2} = \sqrt{\frac{a}{b}}\,\omega$. Checking stability of $\theta_0 = \pi$

$$\left.\frac{\mathrm{d}\ddot{\theta}}{\mathrm{d}\theta}\right|_{\theta=\pi} = -\frac{a}{b}\omega^2 \sin\pi = \frac{a}{b}\omega^2\,, \tag{4.6}$$

so $\theta_0 = \pi$ is unstable. The equation of motion shows that it's like a simple pendulum with $\frac{g}{l} \to \frac{a}{b}\,\omega^2$. In summary, $\boxed{\theta_0 = 0 \text{ is stable with } \omega_0 = \sqrt{\frac{a}{b}}\,\omega}$ and $\boxed{\theta_0 = \pi \text{ is unstable}}$.