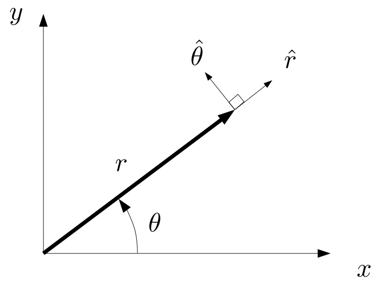
1 plane polar coordinates

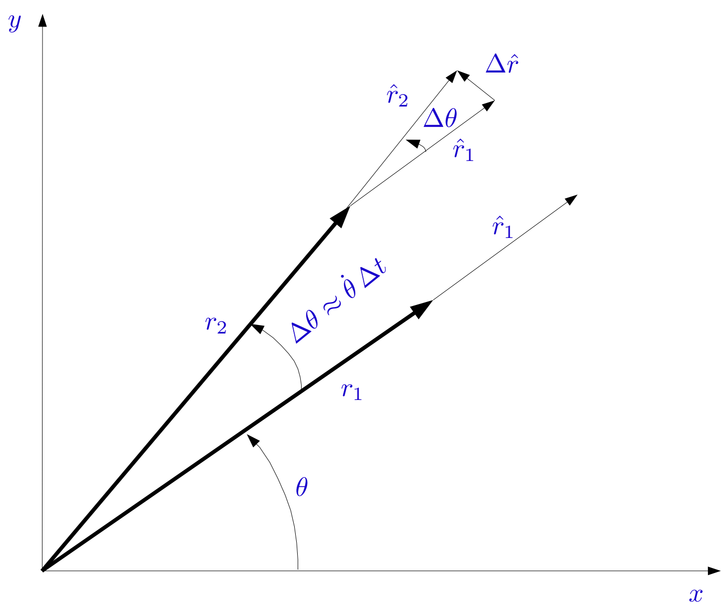
The vector position of a particle in plane polar coordinates can be represented as $\vec{r} = r\hat{r}$, where $\hat{r}$ is the unit vector in the direction of increasing r at a given θ.



1.1 velocity in plane polar coordinates

Show that the velocity of a particle in plane polar coordinates can be represented as $\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$, where $\hat{\theta}$ is the unit vector in the direction of increasing θ. Note: the direction of the $\hat{r}$ and $\hat{\theta}$ change with the value of θ. $\hat{r}$ is written as $\mathbf{e}_r$, and $\hat{\theta}$ is written as $\mathbf{e}_\theta$, in section 1.14 of Thornton and Marion. Your instructor prefers the hat notion because it works on the chalk board, and other reasons.

1.1 solution



For a change in r $\hat{r}$ does not change. From the figure above we see that for a small change θ, we call $\Delta\theta$, that $\hat{r}$ changes like $\Delta\hat{r}$

$$\Delta\hat{r} = \left(2\sin\frac{\Delta\theta}{2}\right)\hat{\theta} \approx 2\frac{\Delta\theta}{2}\hat{\theta} = \Delta\theta\hat{\theta}, \tag{1.1}$$

where we have used the fact that $\frac{\Delta\theta}{2}$ is a small angle. Dividing this by a small change in time Δt gives

$$\frac{\Delta\hat{r}}{\Delta t} \approx \frac{\Delta\theta}{\Delta t}\hat{\theta}. \tag{1.2}$$

We take the limit as $\Delta t \to 0$ giving

$$\dot{\hat{r}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} = \dot{\theta}\hat{\theta}, \tag{1.3}$$

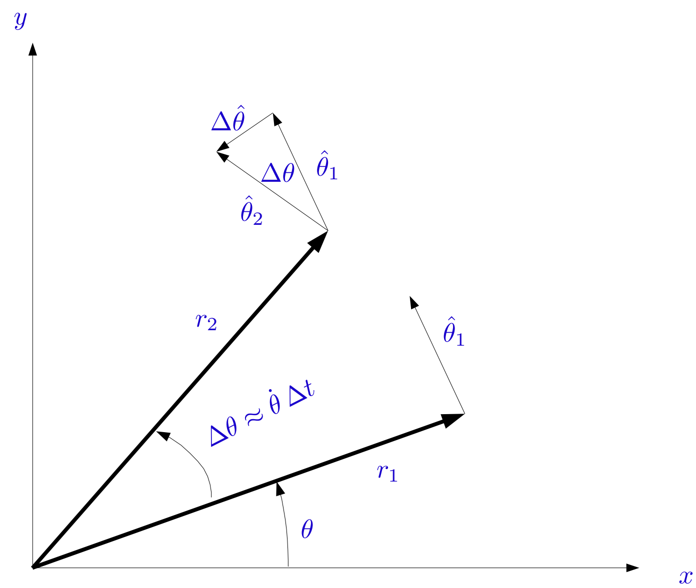
where the $\approx$ becomes $=$ because we have taken the limit $\Delta t \to 0$. So using the chain rule for differentiation on equation 1.1 we get

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{1.4}$$

1.2 acceleration in plane polar coordinates

Find the acceleration, $\ddot{\vec{r}}$, of the particle in plane polar coordinates. Answer in terms of r and θ, and there first and second time derivatives, $\hat{r}$ and $\hat{\theta}$.

1.2 solution



First we will get $\dot{\hat{\theta}}$. $\hat{\theta}$ does not change when r changes. The figure above shows how $\hat{\theta}$ changes with a small change in θ, we call $\Delta\theta$.

$$\Delta\hat{\theta} = \left(2\sin\frac{\Delta\theta}{2}\right)(-\hat{r}) \approx -2\frac{\Delta\theta}{2}\hat{r} = -\Delta\theta\hat{r}. \tag{1.5}$$

Dividing this by a small change in time Δt gives

$$\frac{\Delta\hat{\theta}}{\Delta t} \approx -\frac{\Delta\theta}{\Delta t}\hat{r}. \tag{1.6}$$

We take the limit as Δt goes to zero giving

$$\dot{\hat{\theta}} = -\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{r} = -\dot{\theta}\hat{r}, \tag{1.7}$$

where the $\approx$ becomes $=$ because we have taken the limit $\Delta t \to 0$. Differentiating equation 1.4 with respect to time, using the chain rule, and equations 1.2 and 1.7 gives

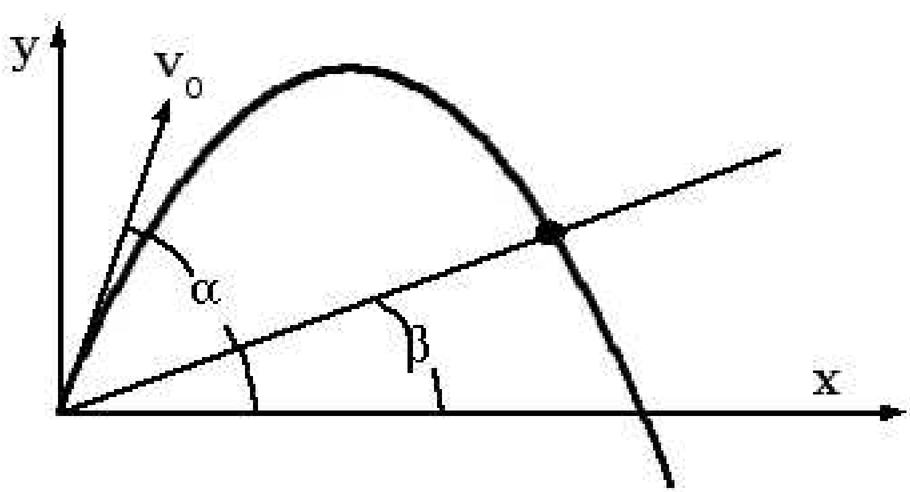
$$\begin{aligned} \ddot{\vec{r}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}\right) && (1.8)\\ &= \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} && (1.9)\\ &= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}\dot{\theta}\hat{r} && (1.10)\\ &= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta} && (1.11) \end{aligned}$$

So we have $\boxed{\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}}$ like in equation 1.98 in the Thornton and Marion text.

2 review - free fall with no air drag

A projectile is fired from the base of an inclined plane up the inclined plane. The initial velocity of the projectile is v_0 at an angle α measured from the horizontal. The angle of slope of the inclined plane is β where, $\beta < \alpha$. Find the time t_1, from when the projectile is fired, for the projectile to impact the inclined plane.

2.0 solution



The equations of motion for the projectile can be written as

$$\begin{aligned} x &= v_{x0}t && (2.1)\\ y &= v_{y0}t - \frac{1}{2}gt^2, && (2.2) \end{aligned}$$

where $v_{x0} = v_0\cos\alpha$, $v_{y0} = v_0\sin\alpha$, and t is time measured from when the projectile was at the origin.

The equation for the line of the inclined plane is

$$y = x\tan\beta. \quad (2.3)$$

Putting together equations 2.1, 2.2, and 2.3 at $t = t_1$ gives

$$\begin{aligned} v_{y0}t_1 - \frac{1}{2}gt_1^2 &= (v_{x0}t_1)\tan\beta && (2.4)\\ \Rightarrow &\quad \left[(v_{y0} - v_{x0}\tan\beta) - \frac{1}{2}gt_1\right]t_1 = 0 && (2.5) \end{aligned}$$

which has the two solutions for t_1,

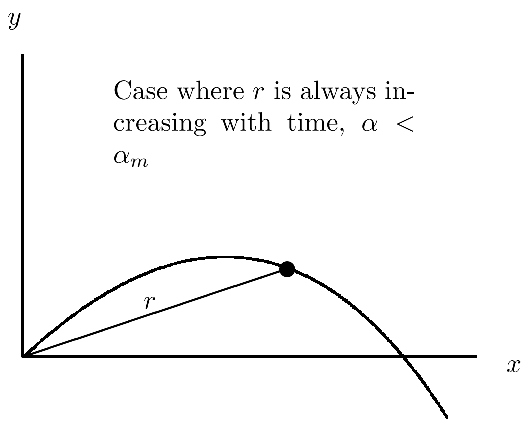
$$t_1 = 0 \tag{2.6}$$
$$t_1 = \frac{2}{g}\left(v_{y0} - v_{x0}\tan\beta\right) = \frac{2v_0}{g}\left(\sin\alpha - \cos\alpha\tan\beta\right). \tag{2.7}$$

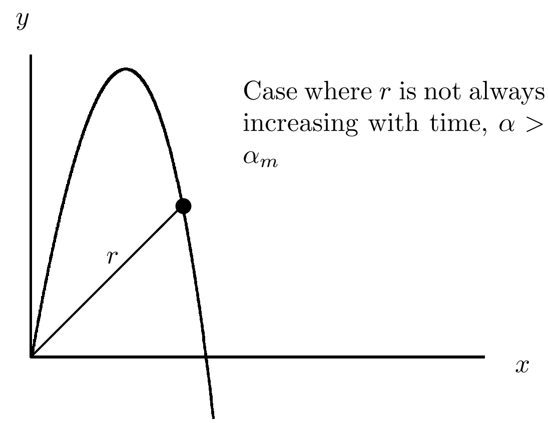
So the solution of interest is $\boxed{t_1 = \frac{2v_0}{g}\left(\sin\alpha - \cos\alpha\tan\beta\right)}$.

3 more free fall with no air drag

3.1

Find the largest angle, α_m, as measured from the horizontal, with which a particle can be projected such that the distance from the launch point to the particle will always be increasing. See the figure below. Note: If, when solving this problem, you find that the statement of this problem is a little inconsistent (as your instructor did), have a look at the next part of this problem.





3.1 solution

Setting the release point to the origin gives the following equations of motion for the projectile

$$x = v_{x0}t \tag{3.1}$$
$$y = v_{y0}t - \frac{1}{2}gt^2, \tag{3.2}$$

where $v_{x0} = v_0\cos\alpha$, $v_{y0} = v_0\sin\alpha$, α is the launch angle as measured from the horizontal, and v_0 is the initial speed of the projectile.

Let r be the distance to the projectile from the point of release. So

$$r^2 = x^2 + y^2 \tag{3.3}$$
$$= \left(v_{x0}t\right)^2 + \left(v_{y0}t - \frac{1}{2}gt^2\right)^2 \tag{3.4}$$
$$= \left(v_{x0}^2 + v_{y0}^2\right)t^2 - v_{y0}gt^3 + \frac{1}{4}g^2t^4 \tag{3.5}$$
$$= v_0^2t^2 - v_{y0}gt^3 + \frac{1}{4}g^2t^4. \tag{3.6}$$

If r is always increasing with time than we must have $\frac{d(r^2)}{dt} > 0$, which from equation 3.6 gives

$$\left(2v_0^2 - 3v_{y0}gt + g^2t^2\right)t > 0 \tag{3.7}$$

$$\Rightarrow \quad g^2t\left(t^2 - 3\frac{v_{y0}}{g}t + 2\frac{v_0^2}{g^2}\right) > 0. \tag{3.8}$$

We can use the quadratic formula to factor the expression in parentheses in 3.8 to give the expression

$$g^2t\left(t - t_+\right)\left(t - t_-\right) > 0, \tag{3.9}$$

where

$$t_\pm = \frac{3\frac{v_{y0}}{g} \pm \sqrt{9\frac{v_{y0}^2}{g^2} - 8\frac{v_0^2}{g^2}}}{2} \tag{3.10}$$

$$= \frac{v_0}{2g}\left(3\sin\alpha \pm \sqrt{9\sin^2\alpha - 8}\right). \tag{3.11}$$

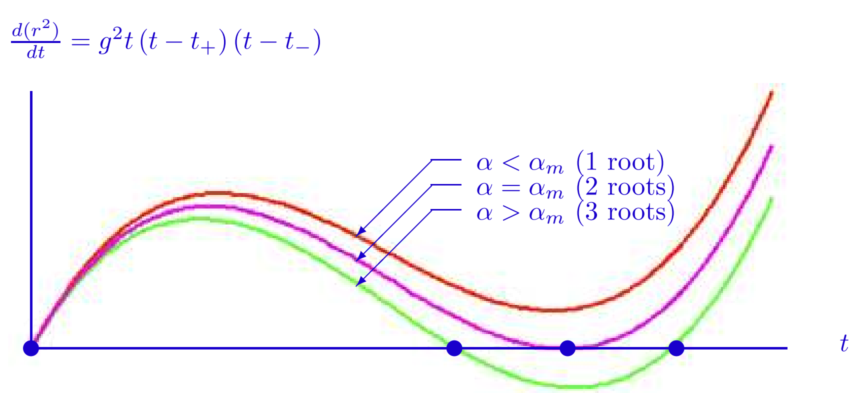
$t_\pm$ from equation 3.11 can only be a solution to expression 3.9 if $9\sin^2\alpha < 8$, so that t_+ and t_- are complex, and complex conjugates of each other. In the limiting case when $9\sin^2\alpha = 8$ the inequality in expressions 3.9 and the limiting angle of interest is $\boxed{\alpha_m = \arcsin\frac{2\sqrt{2}}{3} \simeq 70.53^o}$.

3.2

Explain why your instructor feels that the problem in subsection 3.1, which was restated from Thornton & Marion problem 2-19, should be rewritten as: Find the angle, α_m, as measured from the horizontal, below which a particle can be projected such that the distance from the launch point to the particle will always be increasing for all time after the particle is released.

3.2 solution

Expression 3.8 is satisfied for all t greater than 0 so long as $\sin^2\alpha < \frac{8}{9}$. When $\alpha = \alpha_m$, or $\sin^2\alpha = \frac{8}{9}$, there will be an instant in time, $\sqrt{2}\frac{v_0}{g}$, when $\frac{d(r^2)}{dt}$, and also $\frac{dr}{dt}$, is zero, and so r is not increasing at time $\sqrt{2}\frac{v_0}{g}$, as illustrated in the plot below, and so the distance from the launch point to the particle will not always be increasing when $\alpha = \alpha_m$, but will always be increasing when $\alpha < \alpha_m$. Also at $t = 0$ the distance from the launch point to the particle is not increasing.



4 how high with v^2 air drag?

A ball traveling through a fluid (kind of like air) has a drag force that is proportional to the square of the speed of the ball. Near the surface of the earth the terminal speed of the free falling ball is v_t. The ball is thrown straight up with an initial speed of v_0.

4.1 how high?

Find h, how high from the point of release that the ball will travel. Answer in terms of v_t, v_0, and g, the acceleration due to gravity.

4.1 solution

We will write the ordinary differential equation of motion for the ball, integrate it twice to find first the velocity of the ball $v(t)$ and then the height of the ball $y(t)$, find the time t_1 at which the velocity of the ball is zero, $v(t_1) \equiv 0$, and finally plug that time, t_1, into the height of the ball to give the maximum height of the ball $h = y(t_1)$.

Newton's 2nd law for the ball as it is going up can be written as

$$m\dot{v} = -mg - bv^2, \tag{4.1}$$

where m is the mass of the ball, v is the velocity of the ball $\frac{dy}{dt}$ with the positive y direction being up, and b is the proportionality constant for the fluid drag force.

We can find b as a function of given parameters by the definition of terminal velocity being the speed of the ball when it has been falling a long time and it is traveling at a constant speed of v_t. Newton's 2nd law for the ball when it is traveling at v_t is

$$m\dot{v} = 0 = -mg + bv_t^2, \tag{4.2}$$

where the sign of the drag force is now positive because the ball is moving down, and the drag force is always in the direction opposite to the direction of $\vec{v}$, the velocity of the ball. This gives

$$b = m\frac{g}{v_t^2}. \tag{4.3}$$

So equation 4.1 may now be written as

$$m\dot{v} = -mg - \left(m\frac{g}{v_t^2}\right)v^2 \tag{4.4}$$

$$\Rightarrow \quad \dot{v} = -g - \frac{g}{v_t^2}v^2 \tag{4.5}$$

$$\Rightarrow \quad \frac{\frac{dv}{dt}dt}{g + \frac{g}{v_t^2}v^2} = -dt \tag{4.6}$$

which we solve for $v(t)$ giving

$$\int_{v'=v_0}^{v} \frac{dv'}{g + \frac{g}{v_t^2}v'^2} = -\int_{t'=0}^{t} dt' \tag{4.7}$$

$$\Rightarrow \quad \int_{v'=v_0}^{v} \frac{dv'}{v_t^2 + v'^2} = -\frac{g}{v_t^2}\int_{t'=0}^{t} dt' \tag{4.8}$$

$$\Rightarrow \quad \int_{v'=v_0}^{v} \frac{dv'}{v_t^2 + v'^2} = -\frac{g}{v_t^2}t \tag{4.9}$$

From integral tables we have

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \tag{4.10}$$

which we can use to solve the integral in equation 4.9 to get

$$\frac{1}{v_t}\tan^{-1}\frac{v'}{v_t}\bigg|_{v'=v_0}^{v} = -\frac{g}{v_t^2}t \quad (4.11)$$

$$\Rightarrow \quad \tan^{-1}\frac{v}{v_t} - \tan^{-1}\frac{v_0}{v_t} = -\frac{g}{v_t}t \quad (4.12)$$

$$\Rightarrow \quad v(t) = v_t \tan\left(\tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t\right). \quad (4.13)$$

We can integrate $v(t)$ while setting the initial height to zero giving

$$\begin{aligned} y(t) &= \int_{t'=0}^{t} v(t')dt && (4.14)\\ &= \int_{t'=0}^{t} v_t \tan\left(\tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t'\right)dt'. && (4.15)\end{aligned}$$

From integral tables we have

$$\int \tan ax dx = -\frac{1}{a}\ln\cos ax, \quad (4.16)$$

and with the substitution

$$\begin{aligned} x &= \tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t' && (4.17)\\ dx &= -\frac{g}{v_t}dt' && (4.18)\\ a &= 1 && (4.19)\\ t'=0 \quad &\Rightarrow \quad x = \tan^{-1}\frac{v_0}{v_t} && (4.20)\\ t'=t \quad &\Rightarrow \quad x = \tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t && (4.21)\end{aligned}$$

we have

$$\begin{aligned} y(t) &= -\frac{v_t^2}{g}\int_{x=\tan^{-1}\frac{v_0}{v_t}}^{\tan^{-1}\frac{v_0}{v_t}-\frac{g}{v_t}t} \tan x dx && (4.22)\\ &= -\frac{v_t^2}{g}\left[-\ln(\cos x)\right]\Big|_{x=\tan^{-1}\frac{v_0}{v_t}}^{\tan^{-1}\frac{v_0}{v_t}-\frac{g}{v_t}t} && (4.23)\\ &= -\frac{v_t^2}{g}\left\{\ln\left[\cos\left(\tan^{-1}\frac{v_0}{v_t}\right)\right] - \ln\left[\cos\left(\tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t\right)\right]\right\}. && (4.24)\end{aligned}$$

We find the time at which the ball is at the top of it's path by setting $v(t=t_1)=0$ in equation 4.13 to give

$$\tan^{-1}\frac{v_0}{v_t} - \frac{g}{v_t}t_1 = 0. \quad (4.25)$$

Plugging this value of time $t=t_1$ into $y(t)$ in equation 4.24 gives

$$\begin{aligned} h &= y(t_1) && (4.26)\\ &= -\frac{v_t^2}{g}\left\{\ln\left[\cos\left(\tan^{-1}\frac{v_0}{v_t}\right)\right] - \ln\left[\cos(0)\right]\right\} && (4.27)\\ &= -\frac{v_t^2}{g}\ln\left(\frac{v_t}{\sqrt{v_t^2+v_0^2}}\right). && (4.28)\end{aligned}$$

which gives $\boxed{h = -\frac{v_t^2}{g}\ln\left(\frac{v_t}{\sqrt{v_t^2+v_0^2}}\right)}$.

4.2 limiting case

Show that your result, h, from subsection 4.1 is the same as the case when there is no drag by taking the limit when v_t goes to infinity of the expression that you got for h in subsection 4.1.

4.2 solution

For the case with no drag, with the initial condition $y(t=0)=0$,

$$y(t) = v_0 t - \frac{1}{2} g t^2 \tag{4.29}$$

$$v(t) = v_0 - g t, \tag{4.30}$$

where $v \equiv \frac{dy}{dt}$ and v_0 is the initial value of $\frac{dy}{dt}$. At the top of the path

$$v(t_1) = 0 \tag{4.31}$$

$$= v_0 - g t_1 \tag{4.32}$$

$$\Rightarrow t_1 = \frac{v_0}{g}. \tag{4.33}$$

Plugging $t \to t_1$ into $y(t)$ in equation 4.29 gives h

$$h = y(t_1) \tag{4.34}$$

$$= v_0 \left(\frac{v_0}{g}\right) - \frac{1}{2} g \left(\frac{v_0}{g}\right)^2 \tag{4.35}$$

$$= \frac{1}{2}\frac{v_0^2}{g}. \tag{4.36}$$

We can rewrite equation 4.28 as

$$h = -\frac{v_t^2}{g} \ln\left(\sqrt{\frac{1}{1+\frac{v_0^2}{v_t^2}}}\right). \tag{4.37}$$

We note that when $\frac{v_t}{v_0}$ is large that $\frac{v_0^2}{v_t^2}$ will be small, so we expand the square root ($[1+x]^{-\frac{1}{2}} \approx 1 - \frac{1}{2}x$) in equation about small ($x =$) $\frac{v_0^2}{v_t^2}$ giving

$$h = -\frac{v_t^2}{g} \ln\left(1 - \frac{1}{2}\frac{v_0^2}{v_t^2}\right) \tag{4.38}$$

and expanding the ln ($ln[1+x] \approx x$) about small ($x =$) $\frac{1}{2}\frac{v_0^2}{v_t^2}$ gives

$$h = -\frac{v_t^2}{g}\left(-\frac{1}{2}\frac{v_0^2}{v_t^2}\right) \tag{4.39}$$

$$= \frac{1}{2}\frac{v_0^2}{g} \tag{4.40}$$

which is the same as the height, h, in the case with no drag force in equation 4.36.