This starts with an exercise that walks you through a problem. It's a little different format from the last homework.

A particle of mass m moves in one dimension, r, with the following potential energy

$$U(r) = ar + \frac{b}{r^2}, \tag{0.1}$$

where a and b are positive constants, and the position of the particle, r, is always positive.

1 Force (10 pts)

Find the force, $F(r)$, from this potential as a function of r.

1.0 solution

$$F(r) = -\frac{\partial U(r)}{\partial r} \quad \Rightarrow \quad \boxed{F(r) = -a + \frac{2b}{r^3}} \tag{1.1}$$

2 Equilibrium Position (5 pts)

Find r_0, the one equilibrium r position of the particle as a function of a and b.

2.0 solution

$$F(r_0) = 0 \quad \Rightarrow \quad -a + \frac{2b}{r_0^3} = 0 \quad \Rightarrow \quad \boxed{r_0 = \sqrt[3]{\frac{2b}{a}}} \tag{2.1}$$

3 Scale $U(r)$ (10 pts)

Rewrite $U(r)$ replacing parameters a and b with parameters r_0 and $U_0 \equiv U(r = r_0)$.

3.0 solution

From the definition of U_0 and equation 2.1 we have the two equations

$$U_0 = r_0 a + \frac{1}{r_0^2} b \tag{3.1}$$

$$0 = -a + \frac{2}{r_0^3} b \tag{3.2}$$

which we can solve for a and b in terms of U_0 and r_0 giving

$$a = \frac{2}{3} \frac{U_0}{r_0} \tag{3.3}$$

$$b = \frac{1}{3} U_0 r_0^2 \tag{3.4}$$

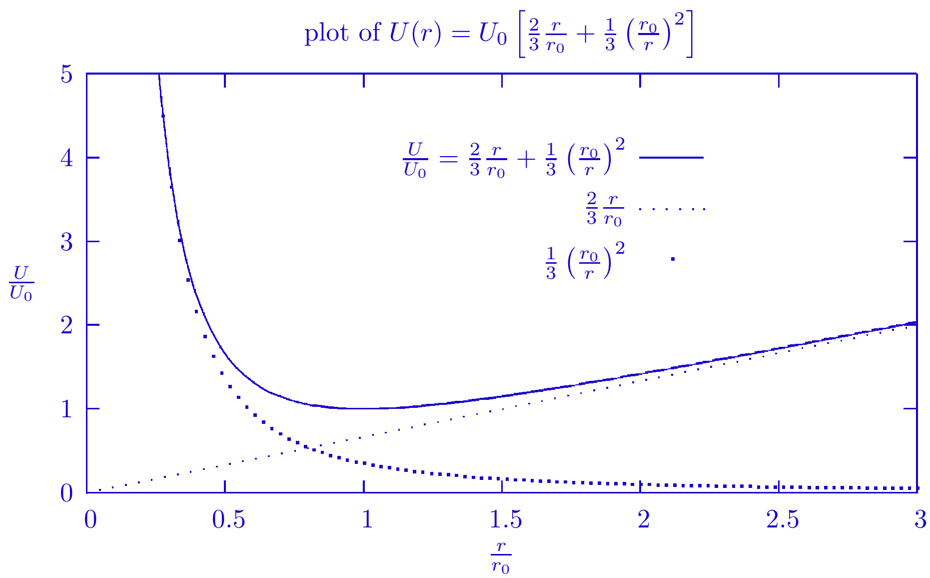
which with equation 0.1 gives

$$\boxed{U(r) = U_0 \left[\frac{2}{3} \frac{r}{r_0} + \frac{1}{3} \left(\frac{r_0}{r} \right)^2 \right]} \tag{3.5}$$

4 Plot $U(r)$ (10 pts)

Note that the shape of the function $U(r)$ does not change with the parameters U_0 and r_0, it just gets scaled along the U-direction with the value of U_0, and along the r-direction with the value of r_0. Make a plot of $\frac{U(r)}{U_0}$ as a function of $\frac{r}{r_0}$.

4.0 solution



5 Expanding about the Equilibrium Position (10 pts)

When the particle is displaced a small amount from r_0 in the positive r direction or the negative r direction it is pushed back to $r = r_0$ by the force from this potential. We call this equilibrium position, $r = r_0$, a stable equilibrium position. The shape of $U(r)$ at, or near, $r = r_0$, is concave up, like a valley.

Expand $U(r)$ as a Taylor series about $r = r_0$ up to, and including, the $(r - r_0)^2$ term. Answer in terms of U_0, r_0, and r. Recall that a Taylor series expansion has the form

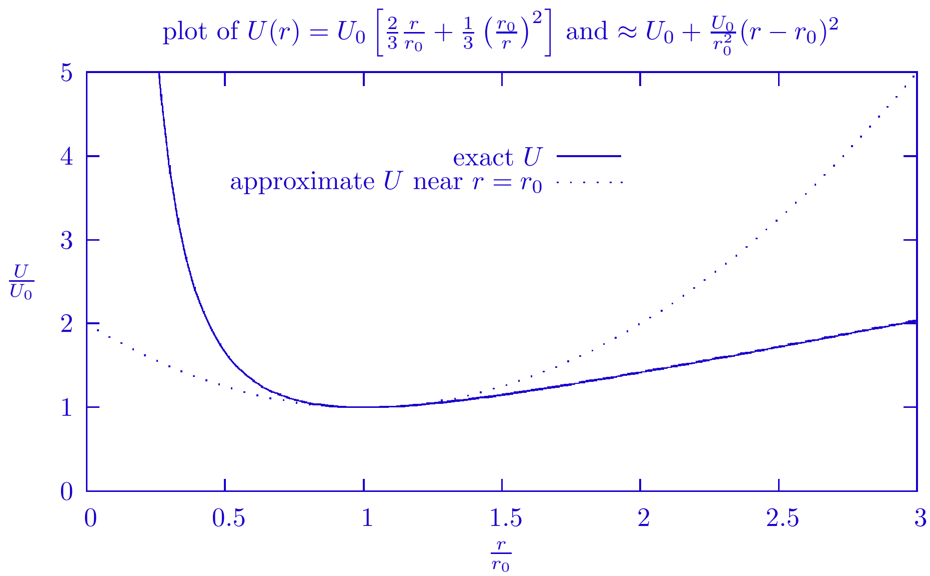
$$U(r) \approx \sum_{n=0}^{N} \frac{(r - r_0)^n}{n!} \left. \frac{\mathrm{d}^n U(r_\star)}{\mathrm{d}r_\star{}^n} \right|^{r_\star = r_0} . \tag{5.1}$$

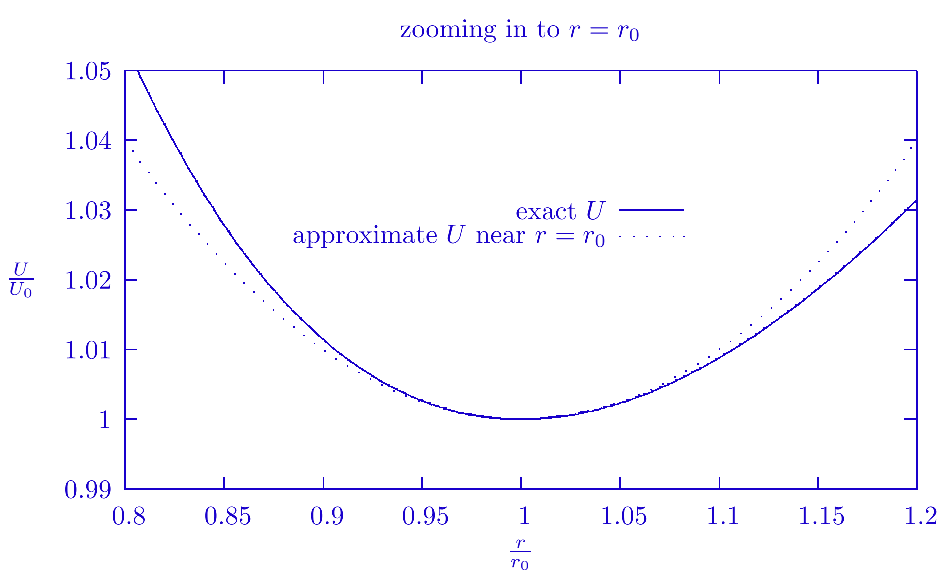
5.0 solution

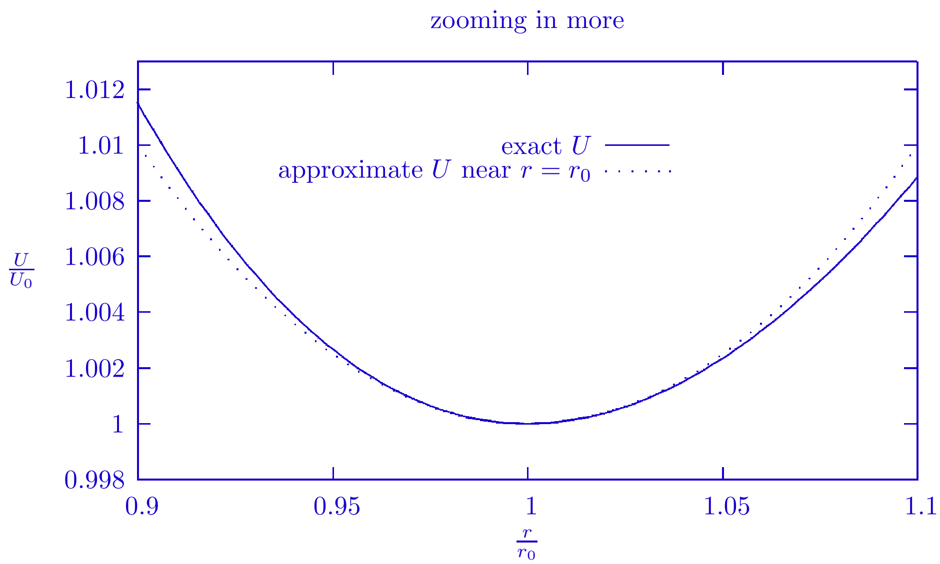
$$U(r) \approx U(r_0) + U'(r_0)\,[r - r_0] + \frac{1}{2}U''(r_0)\,[r - r_0]^2 = U_0 + \frac{U_0}{r_0^2}\,(r - r_0)^2 \tag{5.2}$$

therefore

$$\boxed{U(r) \approx U_0 + \frac{U_0}{r_0^2}\,(r - r_0)^2}. \tag{5.3}$$







6 Small Oscillations about the Equilibrium Position (10 pts)

Note that when $U(r)$ is expanded about $r = r_0$ to $(r - r_0)^2$ it has the same form as the potential for a 1-D simple harmonic oscillator

$$U(x) = C + \frac{1}{2}kx^2, \tag{6.1}$$

where $x = r - r_0$, C is a constant, and k is the spring constant.

6.1

For our potential, $U(r)$, what is the spring constant, k, when we are near $r = r_0$? Answer in terms of constants U_0 and r_0.

6.1 solution

By inspection of equation 5.2 with 6.1

$$\boxed{k = \frac{2U_0}{r_0^2}}. \tag{6.2}$$

6.2

What will be the angular frequency of oscillation, ω_0, of the particle about the equilibrium position $r = r_0$? Express your answer in terms of U_0, r_0 and m. Recall that the angular frequency of oscillation for simple harmonic oscillator is $\omega_0 = \sqrt{\frac{k}{m}}$, where k is the spring force constant and m is the mass.

6.2 solution

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2U_0}{m r_0^2}} \tag{6.3}$$

therefore

$$\boxed{\omega_0 = \sqrt{\frac{2U_0}{m r_0^2}}}. \tag{6.4}$$

7 Another Way (15 pts)

7.1 Equation of Motion (5)

Write the equation of motion of the particle in this 1-D potential. Express your answer in terms of U_0, r_0, m, (without a and b) and r and its time derivatives. So your answer should be of the form $m\ddot{r} = f(r)$, where $f(r)$ is the force as a function of variable r and parameters U_0 and r_0.

7.1 solution

From equations 1.1, 3.3, 3.4 and Newton's 2nd law we have

$$F(r) = -a + \frac{2b}{r^3} \qquad \Rightarrow \qquad \boxed{m\ddot{r} = -\frac{2}{3}\frac{U_0}{r_0} + \frac{2}{3}U_0\frac{r_0^2}{r^3}}. \tag{7.1}$$

7.2 Expand the Equation of Motion about the Equilibrium Position (10)

Expand the equation of motion about the equilibrium position, r_0, by making the substitution $r \equiv r_0 + \eta$, where η is small compared to r_0, and show that the equation of motion of η is that of simple harmonic motion, $\ddot{\eta} = -\omega_0^2\eta$, where ω_0 is the constant angular frequency that is a function of the constant parameters U_0 and r_0. Recall the binomial series expansion $(1+x)^n \approx 1 + nx$ for small x, of course using Taylor series should give the same result.

7.2 solution

$$r = r_0 + \eta \qquad \Rightarrow \qquad \ddot{r} = \ddot{\eta} \tag{7.2}$$

$$\frac{1}{r^3} = \frac{1}{(r_0+\eta)^3} = \frac{1}{{r_0}^3\left(1+\frac{\eta}{r_0}\right)^3} = \frac{1}{{r_0}^3}\left(1+\frac{\eta}{r_0}\right)^{-3} \approx \frac{1}{{r_0}^3}\left(1-3\frac{\eta}{r_0}\right) \tag{7.3}$$

Combining this with the equation of motion (equation 7.1) gives

$$m\,\ddot{\eta} = -\frac{2}{3}\frac{U_0}{r_0} + \frac{2}{3}U_0 r_0^2\left[\frac{1}{{r_0}^3}\left(1-3\frac{\eta}{r_0}\right)\right] \qquad \Rightarrow \qquad m\,\ddot{\eta} = -\frac{2U_0}{r_0^2}\,\eta \qquad \Rightarrow \qquad \boxed{\ddot{\eta} = -\frac{2U_0}{m r_0^2}\,\eta} \tag{7.4}$$

which has a constant angular frequency

$$\boxed{\omega_0 = \sqrt{\frac{2U_0}{mr_0^2}}} \tag{7.5}$$

as before.

8 A Different Problem (15 pts)

A particle, with mass m, is under the influence of a force $F = -kx + k\frac{x^3}{\alpha^2}$, where k and α are constants, and k is positive. Determine the potential $U(x)$, such that $U(0) = 0$. Plot a scaled version of $U(x)$, and discuss the motion. What happens when the total energy (potential plus kinetic) is $E = \frac{1}{4}k\alpha^2$?

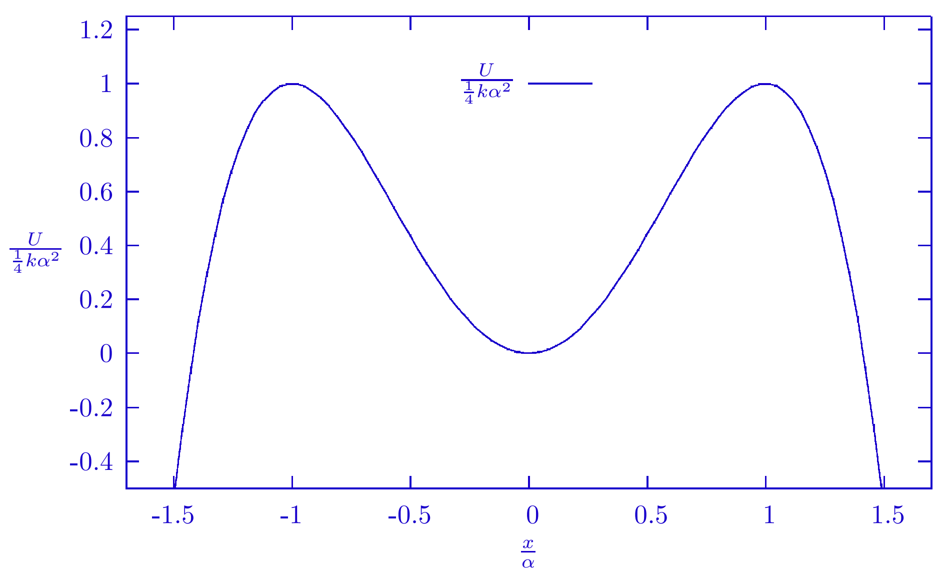
8.0 solution

$$U(x) = -\int \left(-kx + k\frac{x^3}{\alpha^2}\right) \mathrm{d}x \tag{8.1}$$

$$= \frac{1}{2}kx^2 - \frac{1}{4}\frac{k}{\alpha^2}x^4, \tag{8.2}$$

where we have set the constant of integration to zero.

We can plot $\frac{U}{\frac{1}{4}k\alpha^2} = 2\left(\frac{x}{\alpha}\right)^2 - \left(\frac{x}{\alpha}\right)^4$ as a function of $\frac{x}{\alpha}$



We find equilibrium positions from setting $F(x = x_0) = 0$ giving us

$$-kx_0 + k\frac{x_0^3}{\alpha^2} = 0 \tag{8.3}$$

$$\Rightarrow \quad x_0 (x_0 - \alpha)(x_0 + \alpha) = 0 \tag{8.4}$$

$$\Rightarrow \quad x_0 = 0, \pm\alpha. \tag{8.5}$$

We can see from the plot that $x_0 = 0$ is a stable equilibrium position and $x_0 = \pm\alpha$ are unstable equilibrium positions.

The total energy, $E = \frac{1}{2}m\dot{x} + U(x)$, is a constant of the motion. For the case where the total energy E is such that $E > \frac{1}{4}k\alpha^2$ the particle motion will be unbounded for all initial x positions. For the case where the total energy is such that $0 < E < \frac{1}{4}k\alpha^2$ the particle will be pushed to $-\infty$ as time goes to ∞ if it starts with $x < -\alpha$, the particle will be pushed to $+\infty$ as time goes to ∞ if it starts with $x > \alpha$, and the particle motion will be periodic and bounded if it starts with $-\alpha < x < \alpha$. For the case where the total energy is such that $E < 0$,

- a x position where $U > E$ is forbidden,
- if $x > 0$ the motion of x is unbounded going to $+\infty$, and
- if $x < 0$ x goes to $-\infty$ as time goes to ∞.

[*extra*] You may have found that the frequency of small oscillations about $x = 0$, the stable equilibrium position, is $\sqrt{\frac{k}{m}}$. The $k\frac{x^3}{\alpha^2}$ force term causes the frequency to decrease with increasing amplitude.

9 Stability (15 pts)

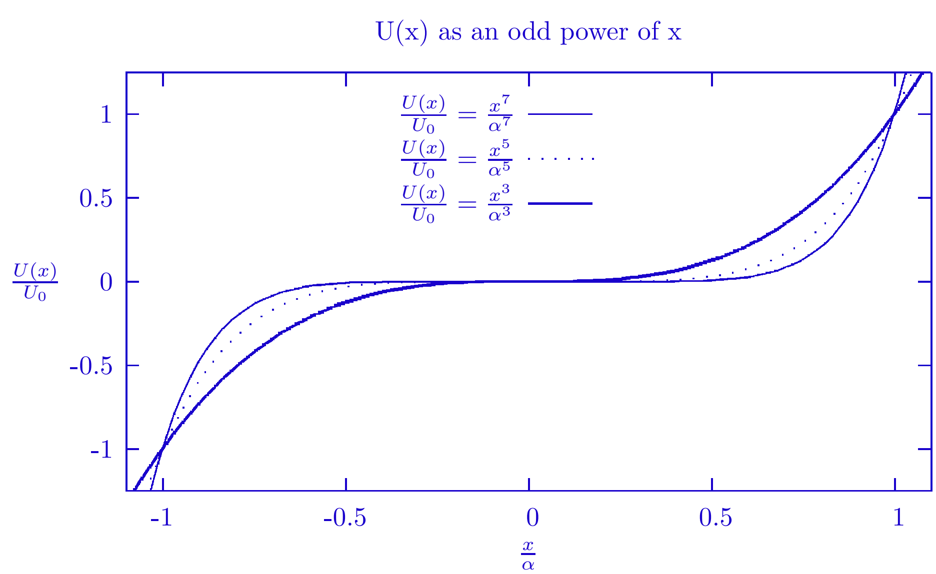
When considering a particle, constrained to move in the x-direction, acted on by a potential $U(x)$, how do we determine when the an equilibrium position x_0, defined by $F_x(x_0) = -\left.\frac{\partial U}{\partial x}\right|^{x=x_0} = 0$, is stable, unstable, or neutral, when $\left.\frac{\partial^n U}{\partial x^n}\right|^{x=x_0} = 0$ for $n = 1$ to m where $m \geq 2$? You may assume that $U(x)$ and all it derivatives are continuous functions.

9.0 solution

We assume that we can expand $U(x)$ in a Taylor series so

$$U(x) \approx \sum_{n=0}^{N} \frac{(x - x_0)^n}{n!} \left.\frac{\mathrm{d}^n U(x_\star)}{\mathrm{d}{x_\star}^n}\right|^{x_\star = x_0}, \tag{9.1}$$

where $N > m$. The stability of U at $x = x_0$ will depend on the first (smallest n) nonzero coefficient in the Taylor series, not including $n = 0$. If the first n is odd the potential $U(x)$ will be unstable at $x = x_0$. The figure below shows a cases there $U \propto x^5$ and $U \propto x^3$. The figure shows how a particle would slide away in the minus x-direction and be contained in the plus x-direction.



If the first nonzero coefficient in the Taylor series has an n that is even the potential $U(x)$ will be stable at $x = x_0$ if the coefficient $\left.\frac{\mathrm{d}^n U(x_\star)}{\mathrm{d}x_\star{}^n}\right|^{x_\star = x_0}$ is greater than zero and unstable if it's less than zero. The figure below shows a case there $U \propto x^6$.

In conclusion, $U(x)$ will be

stable only when the first nonzero derivative of $U(x)$ ($\left.\frac{\mathrm{d}^k U(x_\star)}{\mathrm{d}x_\star{}^k}\right|^{x_\star = x_0} > 0$) is an even number of derivatives (k even) and has a positive value when evaluated at $x = x_0$,

neutrally stable if $U(x)$ is constant and all coefficients in the Taylor series expansion are zero except the $n = 0$ term, and

unstable otherwise.