This starts with an exercise that walks you through a problem. It's a little different format from the last homework.

A particle of mass m moves in one dimension, r, with the following potential energy

$$U(r) = ar + \frac{b}{r^2}, \tag{0.1}$$

where a and b are positive constants, and the position of the particle, r, is always positive.

1 Force (10 pts)

Find the force, $F(r)$, from this potential as a function of r.

2 Equilibrium Position (5 pts)

Find r_0, the one equilibrium r position of the particle as a function of a and b.

3 Scale $U(r)$ (10 pts)

Rewrite $U(r)$ replacing parameters a and b with parameters r_0 and $U_0 \equiv U(r = r_0)$.

4 Plot $U(r)$ (10 pts)

Note that the shape of the function $U(r)$ does not change with the parameters U_0 and r_0, it just gets scaled along the U-direction with the value of U_0, and along the r-direction with the value of r_0. Make a plot of $\frac{U(r)}{U_0}$ as a function of $\frac{r}{r_0}$.

5 Expanding about the Equilibrium Position (10 pts)

When the particle is displaced a small amount from r_0 in the positive r direction or the negative r direction it is pushed back to $r = r_0$ by the force from this potential. We call this equilibrium position, $r = r_0$, a stable equilibrium position. The shape of $U(r)$ at, or near, $r = r_0$, is concave up, like a valley.

Expand $U(r)$ as a Taylor series about $r = r_0$ up to, and including, the $(r - r_0)^2$ term. Answer in terms of U_0, r_0, and r. Recall that a Taylor series expansion has the form

$$U(r) \approx \sum_{n=0}^{N} \frac{(r - r_0)^n}{n!} \left. \frac{\mathrm{d}^n U(r_\star)}{\mathrm{d}{r_\star}^n} \right|^{r_\star = r_0} . \tag{5.1}$$

6 Small Oscillations about the Equilibrium Position (10 pts)

Note that when $U(r)$ is expanded about $r = r_0$ to $(r - r_0)^2$ it has the same form as the potential for a 1-D simple harmonic oscillator

$$U(x) = C + \frac{1}{2}kx^2, \tag{6.1}$$

where $x = r - r_0$, C is a constant, and k is the spring constant.

6.1

For our potential, $U(r)$, what is the spring constant, k, when we are near $r = r_0$? Answer in terms of constants U_0 and r_0.

6.2

What will be the angular frequency of oscillation, ω_0, of the particle about the equilibrium position $r = r_0$? Express your answer in terms of U_0, r_0 and m. Recall that the angular frequency of oscillation for simple harmonic oscillator is $\omega_0 = \sqrt{\frac{k}{m}}$, where k is the spring force constant and m is the mass.

7 Another Way (15 pts)

7.1 Equation of Motion (5)

Write the equation of motion of the particle in this 1-D potential. Express your answer in terms of U_0, r_0, m, (without a and b) and r and its time derivatives. So your answer should be of the form $m\ddot{r} = f(r)$, where $f(r)$ is the force as a function of variable r and parameters U_0 and r_0.

7.2 Expand the Equation of Motion about the Equilibrium Position (10)

Expand the equation of motion about the equilibrium position, r_0, by making the substitution $r \equiv r_0 + \eta$, where η is small compared to r_0, and show that the equation of motion of η is that of simple harmonic motion, $\ddot{\eta} = -\omega_0^2\eta$, where ω_0 is the constant angular frequency that is a function of the constant parameters U_0 and r_0. Recall the binomial series expansion $(1+x)^n \approx 1 + nx$ for small x, of course using Taylor series should give the same result.

8 A Different Problem (15 pts)

A particle, with mass m, is under the influence of a force $F = -kx + k\frac{x^3}{\alpha^2}$, where k and α are constants, and k is positive. Determine the potential $U(x)$, such that $U(0) = 0$. Plot a scaled version of $U(x)$, and discuss the motion. What happens when the total energy (potential plus kinetic) is $E = \frac{1}{4}k\alpha^2$?

9 Stability (15 pts)

When considering a particle, constrained to move in the x-direction, acted on by a potential $U(x)$, how do we determine when the an equilibrium position x_0, defined by $F_x(x_0) = -\frac{\partial U}{\partial x}\Big|^{x=x_0} = 0$, is stable, unstable, or neutral, when $\frac{\partial^n U}{\partial x^n}\Big|^{x=x_0} = 0$ for $n = 1$ to m where $m \geq 2$? You may assume that $U(x)$ and all it derivatives are continuous functions.