1 how small is displacement to make simple harmonic motion

For a large class of forces we can expand the force function into a Taylor series. For example springs have forces that behave like $F = -kx$, where k is a constant and x is the elongation in the spring. This works so long as the value of x is limited in size, but for larger values of x the springs may behave more like $F = -kx - k_3x^3$.

A particle that is constrained to move in one dimension has following net force

$$F(x) = -k_1\,x - k_2\,x^2 - k_3\,x^3, \tag{1.1}$$

where k_1, k_2, and k_3 are positive constants, and x is the displacement of the particle. It is easy to see that $x = 0$ is an equilibrium position, since $F(0) = 0$. And so this is a pretty general form for a 1-D force expanded near equilibrium. (We are excluding discontinuous functions.) No matter what the values of k_1, k_2, and k_3 have we can always have a small enough value of x such that the linear force term is dominant.

How small must x be such that the size of the linear force $-k_1x$ term is 100 or more times the quadratic force term $-k_2x^2$? Also, how small must x be such that the size of the linear force term $-k_1x$ is 100 or more times the cubic force term $-k_3x^3$? Express your answers in terms of k_1, k_2, and k_3.

1.0 solution

Comparing the size of the linear force term and the quadratic force term we get

$$|k_1\,x| \geq 100\,k_2\,x^2 \quad \Rightarrow \quad |x| \leq \frac{1}{100}\frac{k_1}{k_2} \tag{1.2}$$

so as long as $\boxed{|x| \leq \frac{1}{100}\frac{k_1}{k_2}}$ the linear force term will be 100 or more times the quadratic force term.

Comparing the size of the linear force term and the cubic force term we get

$$|k_1\,x| \geq |100\,k_3\,x^3| \quad \Rightarrow \quad |x| \leq \frac{1}{10}\sqrt{\frac{k_1}{k_3}} \tag{1.3}$$

so as long as $\boxed{|x| \leq \frac{1}{10}\sqrt{\frac{k_1}{k_3}}}$ the linear force term will be 100 or more times the cubic force term.

2 energy

A 1-D simple harmonic oscillator consisting of a block and a spring has an frequency $f_0 = 2$Hz. There is no friction or driving force. The block has a mass $m = 100g$. The block is set into motion, with an initial speed of 10 cm/s, from its equilibrium position. Find the maximum potential energy in the spring, $U_{\max}$, and the amplitude, A, of the oscillation. Assume that the potential energy is zero at the equilibrium position. Ignore gravity.

2.0 solution

The total energy E is constant. Let x be the displacement of the oscillator from its equilibrium position. The total energy can be written as $E = \frac{1}{2}m\omega_0^2x^2 + \frac{1}{2}m\dot{x}^2$, where $\omega_o = 2\pi f_0$; therefore the potential energy is a maximum, $U_{\max}$, when $\dot{x} = 0$. So

$$E_i = \frac{1}{2}m\omega_0^2\,(0)^2 + \frac{1}{2}m\dot{x}_i^2 = \tag{2.1}$$

$$= E_f = U_{\max} + \frac{1}{2}m\,(0)^2\,, \tag{2.2}$$

where we defined E_i as the initial energy in the oscillator, and E_f as the energy when the oscillator as zero velocity, which gives

$$U_{\max} = \frac{1}{2} m\dot{x}_i^2 = \frac{1}{2}\,(100g)\,\frac{1kg}{1000g}\left[(10\text{cm/s})\,\frac{1\text{m}}{100\text{cm}}\right]^2 = 0.5\,\text{mJoul}. \tag{2.3}$$

$$\boxed{U_{\max} = 0.5\,\text{mJoul}} \tag{2.4}$$

We have

$$\frac{1}{2} m\omega_0^2 A^2 = E = U_{\max} \tag{2.5}$$

so

$$A = \frac{1}{\omega_0}\sqrt{\frac{2U_{\max}}{m}} = \frac{1}{\omega_0}\sqrt{\frac{2\frac{1}{2}m\dot{x}_i^2}{m}} = \frac{\dot{x}_i}{\omega_0} = \frac{10cm/s}{2\pi(\text{rad/cycle})(2\text{cycle/s})} \approx 0.796\,\text{cm}. \tag{2.6}$$

$$\boxed{A \approx 7.96\,\text{mm}} \tag{2.7}$$