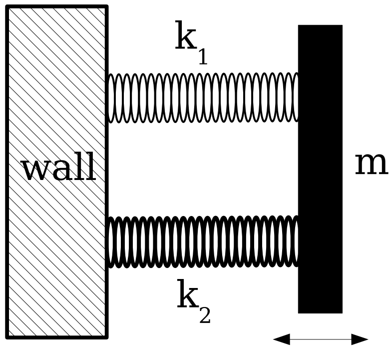
1 two springs

1.1 in parallel

Two massless springs, with spring constants k_1 and k_2, are connected as shown in the figure below. Both springs have the same rest (zero force) length. The wall to the left of the springs does not move. Ignore gravity.



The mass, with mass m, oscillates without friction along the direction shown. The mass is constrained to move only in the direction shown. Find the angular frequency of oscillation, ω_0, of the mass.

1.1 solution

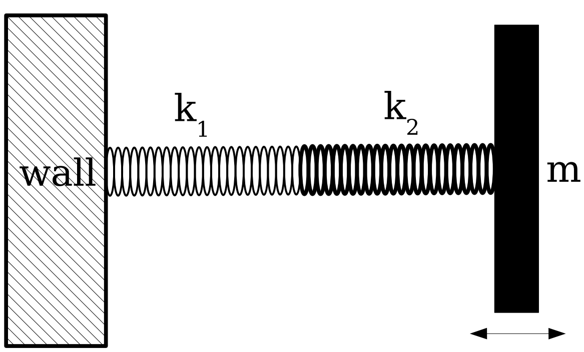
Let x be the displacement of the mass from equilibrium, where positive x is to the right. Applying Newton's 2nd law to the mass gives

$$\sum F_x = -k_1 x - k_2 x = m\ddot{x} \qquad \Rightarrow \qquad \ddot{x} = -\left(\frac{k_1 + k_2}{m}\right) x \tag{1.1}$$

which is the second order differential equation for simple harmonic motion with angular frequency $\boxed{\omega_0 = \sqrt{\frac{k_1 + k_2}{m}}}$.

1.2 in series

Two massless springs with spring constants k_1 and k_2 are connected as shown in the figure below. The wall to the left of the springs does not move. Ignore gravity.



The mass, with mass m, oscillates without friction along the direction shown. Find the angular frequency of oscillation, ω_0, of the mass.

1.2 solution

Let x be the displacement of the mass from equilibrium, where positive x is to the right. Let x_1 be the stretch in the k_1 spring, and x_2 be the stretch in the k_2 spring. So

$$x_1 + x_2 = x. \tag{1.2}$$

Since the springs are massless the forces between the two springs must be balanced so that

$$k_1 x_1 - k_2 x_2 = 0. \tag{1.3}$$

Since only the k_2 spring is pushing and pulling on the mass, applying Newton's 2nd law to the mass gives

$$\sum F_x = -k_2 x_2 = m\ddot{x} \tag{1.4}$$

We need to replace x_2 in equation 1.4 with a function of k_1, k_2, and x using equations 1.2 and 1.3. Solving equations 1.2 and 1.3 for x_2 gives

$$x_2 = \frac{x}{1 + \frac{k_2}{k_1}}. \tag{1.5}$$

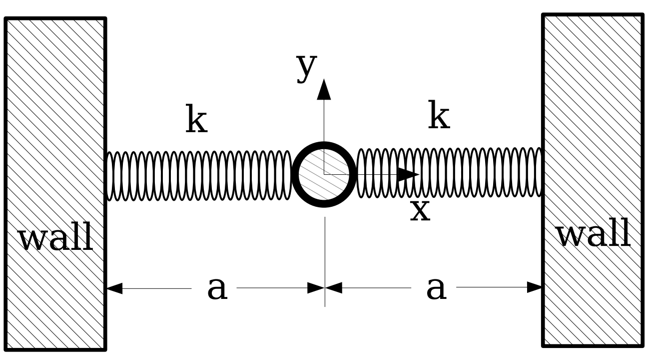
Putting x_2 from equation 1.5 into equation 1.4 gives

$$m\ddot{x} = -k_2 \frac{x}{1 + \frac{k_2}{k_1}} = -\frac{k_1 k_2}{k_1 + k_2} x \tag{1.6}$$

$$\Rightarrow \qquad \ddot{x} = -\frac{k_1 k_2}{m(k_1 + k_2)} x \tag{1.7}$$

which is the second order differential equation for simple harmonic motion with angular frequency $\boxed{\omega_0 = \sqrt{\frac{k_1 k_2}{m(k_1 + k_2)}}}$.

2 two springs in 2-D *(extra credit: 10 pts)*

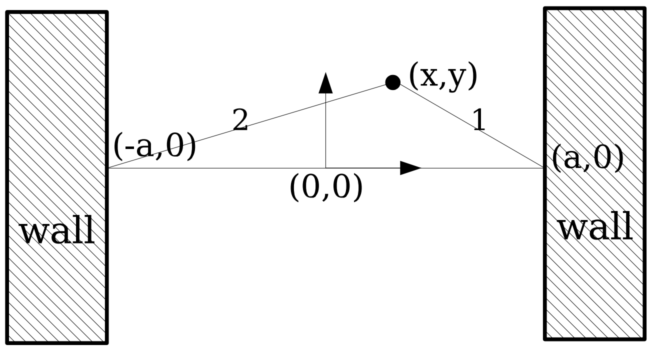


The two linear springs shown have rest length l and spring constant k. l is not necessarily related to the length a shown in the figure. A particle is attached to both strings at the origin of the x-y coordinate system shown. The particle can be moved in the x-y plane while the two springs stay attached pushing and/or pulling on it. The ends of the springs stay attached to the walls, and the walls do not move. You may consider the particle to have negligible size so that the springs have a length a when the particle is at the origin. There are no forces from gravity.

2.1 potential

Find the potential energy in the springs $U(x, y)$ as a function of variables x and y, and parameters k, a, and l.

2.1 solution



$$U = U_1 + U_2 \tag{2.1}$$
$$= \frac{1}{2}k\,(\text{stretch1})^2 + \frac{1}{2}k\,(\text{stretch2})^2 \tag{2.2}$$
$$= \frac{1}{2}k\left[\sqrt{(a-x)^2+y^2} - l\right]^2 + \frac{1}{2}k\left[\sqrt{(a+x)^2+y^2} - l\right]^2 \tag{2.3}$$

so

$$\boxed{U(x,y) = \frac{1}{2}k\left[\sqrt{(a-x)^2+y^2} - l\right]^2 + \frac{1}{2}k\left[\sqrt{(a+x)^2+y^2} - l\right]^2} \tag{2.4}$$

2.2 potential for small x and y

Expand $U(x, y)$ about small x and y to show that this potential is that of a 2-D anisotropic harmonic oscillator that oscillates independently in the x and y directions, and so the potential energy can be written in the form $U(x, y) = \text{constant} + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2$, where k_x and k_y are two different constants.

2.2 solution

Something can only be small when compared to something else. We guess that small x and y means that $x \ll a$ and $y \ll a$. If that works than we'll get reasonable results. In equation 2.4 we first expand a squared binomial and then use the expansion formula $(1+\epsilon)^{\frac{1}{2}} \approx 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2$ keeping terms to size of order $\frac{x^2}{a^2}$ and $\frac{y^2}{a^2}$

$$
\begin{aligned}
U &= \frac{1}{2}ka^2\left[\sqrt{\left(1-\frac{x}{a}\right)^2+\frac{y^2}{a^2}}-\frac{l}{a}\right]^2+\frac{1}{2}ka^2\left[\sqrt{\left(1+\frac{x}{a}\right)^2+\frac{y^2}{a^2}}-\frac{l}{a}\right]^2 && (2.5)\\
&= \frac{1}{2}ka^2\left[\sqrt{\left(1-2\frac{x}{a}+\frac{x^2}{a^2}\right)+\frac{y^2}{a^2}}-\frac{l}{a}\right]^2+\frac{1}{2}ka^2\left[\sqrt{\left(1+2\frac{x}{a}+\frac{x^2}{a^2}\right)+\frac{y^2}{a^2}}-\frac{l}{a}\right]^2 && (2.6)\\
&= \frac{1}{2}ka^2\left[\sqrt{1+\underbrace{\left(-2\frac{x}{a}+\frac{x^2}{a^2}+\frac{y^2}{a^2}\right)}_{\epsilon}}-\frac{l}{a}\right]^2+\frac{1}{2}ka^2\left[\sqrt{1+\underbrace{\left(2\frac{x}{a}+\frac{x^2}{a^2}+\frac{y^2}{a^2}\right)}_{\epsilon}}-\frac{l}{a}\right]^2 && (2.7)\\
&\approx \frac{1}{2}ka^2\left[1+\frac{1}{2}\left(-2\frac{x}{a}+\frac{x^2}{a^2}+\frac{y^2}{a^2}\right)-\frac{1}{8}\left(-2\frac{x}{a}\right)^2-\frac{l}{a}\right]^2\\
&\quad +\frac{1}{2}ka^2\left[1+\frac{1}{2}\left(2\frac{x}{a}+\frac{x^2}{a^2}+\frac{y^2}{a^2}\right)-\frac{1}{8}\left(2\frac{x}{a}\right)^2-\frac{l}{a}\right]^2 && (2.8)\\
&\approx \frac{1}{2}ka^2\left(1-\frac{l}{a}-\frac{x}{a}+\frac{1}{2}\frac{y^2}{a^2}\right)^2+\frac{1}{2}ka^2\left(1-\frac{l}{a}+\frac{x}{a}+\frac{1}{2}\frac{y^2}{a^2}\right)^2 && (2.9)\\
&\approx ka^2\left(1-\frac{l}{a}\right)^2+ka^2\frac{x^2}{a^2}+ka^2\left(1-\frac{l}{a}\right)\frac{y^2}{a^2} && (2.10)\\
&= \text{constant}+kx^2+k\left(1-\frac{l}{a}\right)y^2 && (2.11)\\
& && (2.12)
\end{aligned}
$$

which gives

$$\boxed{U(x,y) = \text{constant} + \frac{1}{2}(2k)x^2 + \frac{1}{2}\left[2k\left(1-\frac{l}{a}\right)\right]y^2}. \quad (2.13)$$

So $\boxed{k_x = 2k \text{ and } k_y = 2k\left(1-\frac{l}{a}\right)}$.

2.3 cases

What happens to the potential energy when $l < a$, $l > a$, and $l = a$?

2.3 solution

For

l < a x and y oscillate independently about a stable equilibrium position $(x = 0, y = 0)$ with effective spring constants $k_x = 2\,k$ and $k_y = 2\,k\left(1-\frac{l}{a}\right)$, with

l > a the x motion is still oscillatory, but the y equilibrium position $y = 0$ is an unstable equilibrium position because U is concave up at $y = 0$, and there are stable y positions that are equal amounts above and below $y = 0$, and with

l = a the x motion is still oscillatory, and in the y direction U is not quadratic any more. The U expansion must be carried out to higher powers in $\frac{y}{a}$ giving an anharmonic oscillation in the y direction, even for small y displacements.

2.4 simple harmonic case

Given that m is the mass of the particle, $l < a$, and both x and y are small, what is the solution to the motion, $x(t)$ and $y(t)$, where t is time, when the initial position is at $x = X_i$ and $y = Y_i$, and the particle starts at rest (not moving)?

2.4 solution

For small x and y

$$U(x,y) = \text{constant} + kx^2 + k\left(1 - \frac{l}{a}\right)y^2. \tag{2.14}$$

Finding the forces and applying Newton's 2nd law gives

$$\sum F_x = -\frac{\partial U}{\partial x} = -2\,k\,x = m\ddot{x} \tag{2.15}$$

$$\sum F_y = -\frac{\partial U}{\partial y} = -2\,k\left(1 - \frac{l}{a}\right)\,y = m\ddot{y} \tag{2.16}$$

which gives two independent simple harmonic motions with the angular frequencies

$$\omega_x = \sqrt{\frac{2k}{m}} \tag{2.17}$$

$$\omega_y = \sqrt{\frac{2k\left(1 - \frac{l}{a}\right)}{m}}, \tag{2.18}$$

where ω_x is the angular frequency for the motion in the x direction and ω_y is the angular frequency for the motion in the y direction. The constants of integration can be found by inspection giving the solutions

$$\boxed{x(t) = X_i \cos\left(\sqrt{\frac{2k}{m}}t\right)} \tag{2.19}$$

$$\boxed{y(t) = Y_i \cos\left(\sqrt{\frac{2k\left(1 - \frac{l}{a}\right)}{m}}t\right)}, \tag{2.20}$$

where we see that $x(0) = X_i$ and $y(0) = Y_i$.

3 two solution forms

The general solution of an undriven, under-damped simple harmonic oscillator $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ can be written as $x = Ce^{-\beta t}\cos(\phi + \delta)$, where $\phi = \sqrt{\omega_0^2 - \beta^2}\,t$, and C and δ are constants of integration. The general solution can also be written as $x = Ae^{-\beta t}\cos\phi + Be^{-\beta t}\sin\phi$, where ϕ is the same as before, and A and B are constants of integration.

Show that these two general solutions can represent the same functions by finding $C(A,B)$ as a function of A and B, and $\delta(A,B)$ as a function of A and B, whereby showing that $C(A,B)e^{-\beta t}\cos(\phi + \delta(A,B)) = Ae^{-\beta t}\cos\phi + Be^{-\beta t}\sin\phi$

3.0 solution

We start with

$$Ce^{-\beta t}\cos(\phi+\delta) = Ae^{-\beta t}\cos\phi + Be^{-\beta t} sin\phi \tag{3.1}$$

and using the trigonometric identity $\cos(a+b) = \cos a \cos b - \sin a \sin b$ in the left hand side, with $a = \phi$ and $b = \delta$, gives

$$Ce^{-\beta t}\cos\phi\cos\delta - Ce^{-\beta t}\sin\phi\sin\delta = Ae^{-\beta t}\cos\phi + Be^{-\beta t}\sin\phi. \tag{3.2}$$

Because this equation must be true of all values of ϕ, the terms with $\cos\phi$, and the terms with $\sin\phi$ must equate independently giving the two separate equations

$$Ce^{-\beta t}\sin\phi\sin\delta = -Be^{-\beta t}\sin\phi \tag{3.3}$$

$$Ce^{-\beta t}\cos\phi\cos\delta = Ae^{-\beta t}\cos\phi. \tag{3.4}$$

Dividing out common factors gives

$$C\sin\delta = -B \tag{3.5}$$

$$C\cos\delta = A. \tag{3.6}$$

So now we may solve the two equations for the two unknowns C and δ in terms of a and b. Summing the square of equation 3.5 to the square of equation 3.6 gives

$$C^2 = A^2 + B^2, \tag{3.7}$$

and dividing equation 3.5 by equation 3.6 gives

$$\tan\delta = -\frac{B}{A} \;\Rightarrow\; \tan(-\delta) = \frac{B}{A} \;\Rightarrow\; -\delta = \tan^{-1}\frac{B}{A} \tag{3.8}$$

So we now have $\boxed{C(A,B) = \sqrt{A^2+B^2}}$ and $\boxed{\delta(A,B) = -\tan^{-1}\left(\frac{B}{A}\right)}$, and therefore

$$\boxed{\sqrt{A^2+B^2}e^{-\beta t}\cos\left[\phi - \tan^{-1}\left(\frac{B}{A}\right)\right] = Ae^{-\beta t}\cos\phi + Be^{-\beta t} sin\phi}. \tag{3.9}$$

4 plotting a damped simple harmonic oscillator

A damped simple harmonic oscillator is modeled by

$$\ddot{x} = -\omega_0^2 x - b\dot{x}, \tag{4.1}$$

where $\omega = 2$rad/s and $b = 0.4$/s, and x is the dependent variable that depends on t time. The oscillator has the initial conditions $x(0) = 1$m and $\dot{x}(0) = \frac{1}{2}$m/s.

With these initial conditions, use your computer to plot $x(t)$ and $\dot{x}(t)$ verses t (time) for $t = 0$ to 10 seconds. Also make a phase plot of $\dot{x}(t)$ verses $x(t)$ for $t = 0$ to 30 seconds.

You may use any 2-D plotter that you like, so long as you can manage to produce a paper copy for your solution to this problem. Gnuplot is a free 2-D plotter that works on GNU/Linux, MS Windows, Mac, and more. You can get gnuplot at http://gnuplot.sourceforge.net/. The instructor used gnuplot to make the solutions to this homework. Another possibility is to use Mathematica.

4.0 solution

The general solution to 4.1 is

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta), \tag{4.2}$$

where $\beta = \frac{b}{2}$, $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and A and δ are constants of integration. Using the initial conditions gives

$$x(0) = A\cos\delta \tag{4.3}$$

$$\dot{x}(0) = -A\beta\cos\delta - A\omega_1\sin\delta. \tag{4.4}$$

Plugging equation 4.3 into the first term on the right side of equation 4.4 gives

$$\dot{x}(0) = -\beta x(0) - \omega_1 A sin\delta \;\Rightarrow\; \frac{\dot{x}(0) + \beta x(0)}{\omega_1} = -A sin\delta. \tag{4.5}$$

We can square and add equations 4.3 and 4.5 to give

$$A^2 = \left(\frac{\dot{x}(0) + \beta x(0)}{\omega_1}\right)^2 + x(0)^2 \;\Rightarrow\; A = \sqrt{\left(\frac{\dot{x}(0) + \beta x(0)}{\omega_1}\right)^2 + x(0)^2}. \tag{4.6}$$

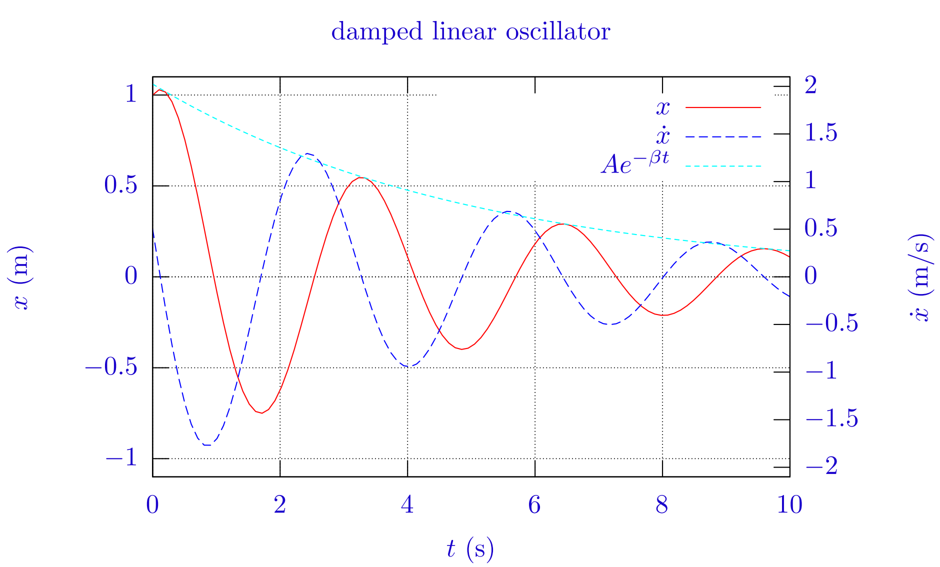
Given A we can solve for δ from equation 4.3 giving

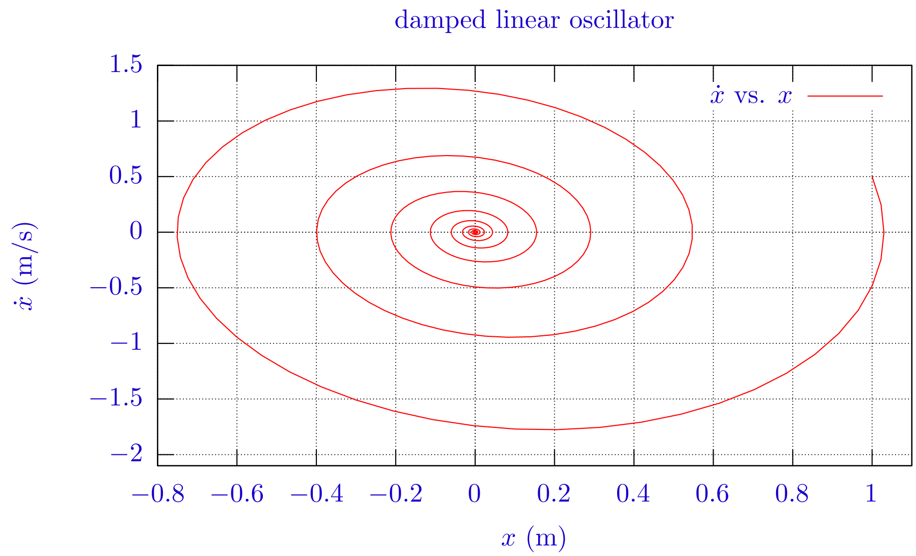
$$\delta = \cos^{-1}\left(\frac{x(0)}{A}\right) \tag{4.7}$$

From $\dot{x}(t)$ differentiating $x(t)$ in equation 4.2 we get

$$\dot{x}(t) = -Ae^{-\beta t}\left[\beta\cos(\omega_1 t - \delta) + \omega_1\sin(\omega_1 t - \delta)\right] = -Ae^{-\beta t}\sqrt{\beta^2 + \omega_1^2}\cos\left[\omega_1 t - \delta - \tan^{-1}\left(\frac{\omega_1}{\beta}\right)\right] \tag{4.8}$$

$$= -A\omega_0 e^{-\beta t}\cos\left[\omega_1 t - \delta - \tan^{-1}\left(\sqrt{\left(\frac{\omega_0}{\beta}\right)^2 - 1}\right)\right]. \tag{4.9}$$





The gnuplot plotting files can be downloaded from the same place this document came from in the two files

- hw04_linOsc.gp and
- hw04_linOscPhase.gp.

5 two first order linear differential equations

The undriven over-damped harmonic oscillator

$$\ddot{x} = -\omega_0^2 x - 2\beta\dot{x}, \tag{5.1}$$

where ω_0 and β are constant parameters, and $\omega_2 \equiv \sqrt{\beta^2 - \omega_0^2}$ is a real constant that is greater than zero, can be rewritten as the two first order linear differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{5.2}$$

where x and $p \equiv \dot{x}$ are the two time dependent variables of the system.

Solve 5.2 finding the two eigenvalues and eigenvectors for this system along the way. Use the form of solution: $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \equiv \vec{a}e^{\lambda t}$, where $\vec{a}$ is an eigenvector and λ is an eigenvalue. Normalizing the eigenvectors is not required. Compare your two eigenvectors with the two dotted lines in the phase plot (with $p \equiv v$) in figure 3-11 (page 115) in Thornton and Marion. You can get a copy of it in the file TandM_figure3.11.pdf where you got this file.

5.0 solution

We substitute $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \vec{a}e^{\lambda t}$ into equation 5.2 giving

$$\lambda \vec{a} e^{\lambda t} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{pmatrix} \vec{a} e^{\lambda t} \tag{5.3}$$

which must be valid to all t, so we can divide out the $e^{\lambda t}$, and we introduce an identity matrix on the left-hand-side giving

$$\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \vec{a} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{pmatrix} \vec{a} \Rightarrow \begin{pmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda - 2\beta \end{pmatrix} \vec{a} = 0. \tag{5.4}$$

In order for there to be a nontrivial solution for the value of $\vec{a}$ in equation 5.4 the determinant of the matrix must be zero, like so

$$\begin{vmatrix} -\lambda & 1 \\ -\omega_0^2 & -\lambda - 2\beta \end{vmatrix} = 0. \tag{5.5}$$

This gives the characteristic equation

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \tag{5.6}$$

which we can solve for λ giving the two eigenvalues

$$\lambda_\pm = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta \pm \omega_2. \tag{5.7}$$

We put the eigenvalues into equation 5.4 giving two dependent equations for our eigenvector components a_1 and a_2, the first of which is

$$-(-\beta \pm \omega_2)a_1 + a_2 = 0. \tag{5.8}$$

So the eigenvectors may be written as

$$\vec{a}_+ = \begin{pmatrix} 1 \\ -\beta + \omega_2 \end{pmatrix} \tag{5.9}$$

$$\vec{a}_- = \begin{pmatrix} 1 \\ -\beta - \omega_2 \end{pmatrix} \tag{5.10}$$

and so the general solution to equation 5.2 is

$$\boxed{\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -\beta + \omega_2 \end{pmatrix} e^{-(\beta - \omega_2)t} + A_2 \begin{pmatrix} 1 \\ -\beta - \omega_2 \end{pmatrix} e^{-(\beta + \omega_2)t}} \tag{5.11}$$

where A_1 and A_2 are the constants of integration.

In comparing with the two dotted lines in the phase plot (with $p \equiv v$) in figure 3-11 (page 115) in Thornton and Marion; we can get two parametric equations of the lines along each eigenvector $\vec{a}$

$$\begin{pmatrix} x \\ p \end{pmatrix} = \tau \vec{a}\,, \tag{5.12}$$

where we introduced τ as the parametric variable. This gives us the following equations for the lines

$$\vec{a}_+ \text{ slow line } \begin{cases} x = \tau(1) \\ p = \tau(-\beta + \omega_2) \end{cases} \tag{5.13}$$

$$\vec{a}_- \text{ fast line } \begin{cases} x = \tau(1) \\ p = \tau(-\beta - \omega_2) \end{cases} \tag{5.14}$$

which we may rewrite by eliminating τ giving

$$\text{slow line} \quad \{p = -(\beta - \omega_2)x \tag{5.15}$$

$$\text{fast line} \quad \{p = -(\beta + \omega_2)x \tag{5.16}$$

as shown in figure 3-11 (page 115) in Thornton and Marion (with $p \equiv v$).