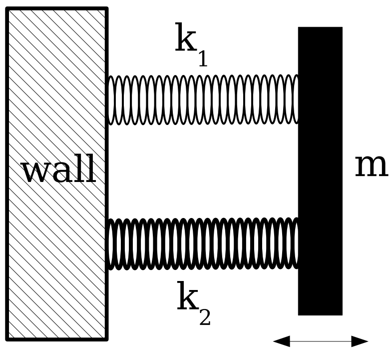
1 two springs

1.1 in parallel

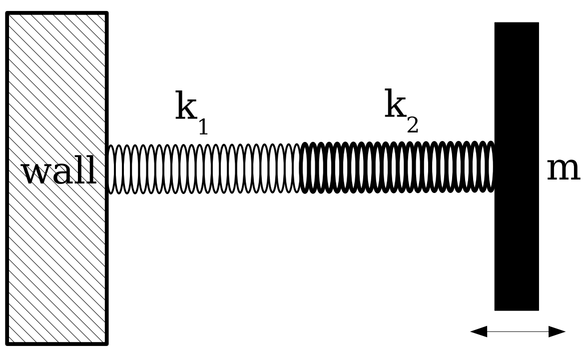
Two massless springs, with spring constants k_1 and k_2, are connected as shown in the figure below. Both springs have the same rest (zero force) length. The wall to the left of the springs does not move. Ignore gravity.



The mass, with mass m, oscillates without friction along the direction shown. The mass is constrained to move only in the direction shown. Find the angular frequency of oscillation, ω_0, of the mass.

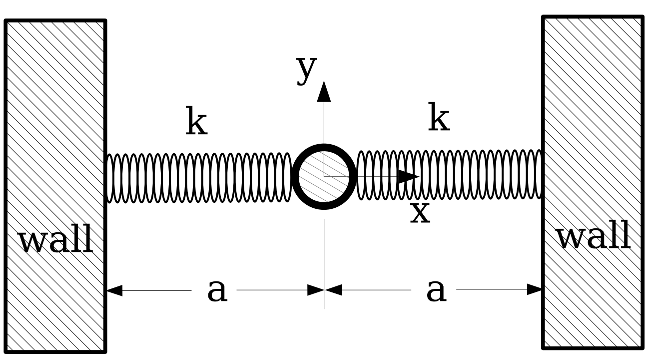
1.2 in series

Two massless springs with spring constants k_1 and k_2 are connected as shown in the figure below. The wall to the left of the springs does not move. Ignore gravity.



The mass, with mass m, oscillates without friction along the direction shown. Find the angular frequency of oscillation, ω_0, of the mass.

2 two springs in 2-D *(extra credit: 10 pts)*



The two linear springs shown have rest length l and spring constant k. l is not necessarily related to the length a shown in the figure. A particle is attached to both strings at the origin of the x-y coordinate system shown. The particle can be moved in the x-y plane while the two springs stay attached pushing and/or pulling on it. The ends of the springs stay attached to the walls, and the walls do not move. You may consider the particle to have negligible size so that the springs have a length a when the particle is at the origin. There are no forces from gravity.

2.1 potential

Find the potential energy in the springs $U(x, y)$ as a function of variables x and y, and parameters k, a, and l.

2.2 potential for small x and y

Expand $U(x, y)$ about small x and y to show that this potential is that of a 2-D anisotropic harmonic oscillator that oscillates independently in the x and y directions, and so the potential energy can be written in the form $U(x, y) = \text{constant} + \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2$, where k_x and k_y are two different constants.

2.3 cases

What happens to the potential energy when $l < a$, $l > a$, and $l = a$?

2.4 simple harmonic case

Given that m is the mass of the particle, $l < a$, and both x and y are small, what is the solution to the motion, $x(t)$ and $y(t)$, where t is time, when the initial position is at $x = X_i$ and $y = Y_i$, and the particle starts at rest (not moving)?

3 two solution forms

The general solution of an undriven, under-damped simple harmonic oscillator $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ can be written as $x = Ce^{-\beta t}\cos(\phi + \delta)$, where $\phi = \sqrt{\omega_0^2 - \beta^2}\, t$, and C and δ are constants of integration. The general solution can also be written as $x = Ae^{-\beta t}\cos\phi + Be^{-\beta t}\sin\phi$, where ϕ is the same as before, and A and B are constants of integration.

Show that these two general solutions can represent the same functions by finding $C(A, B)$ as a function of A and B, and $\delta(A, B)$ as a function of A and B, whereby showing that $C(A, B)e^{-\beta t}\cos(\phi + \delta(A, B)) = Ae^{-\beta t}\cos\phi + Be^{-\beta t}\sin\phi$

4 plotting a damped simple harmonic oscillator

A damped simple harmonic oscillator is modeled by

$$\ddot{x} = -\omega_0^2 x - b\dot{x}, \tag{4.1}$$

where $\omega = 2\text{rad/s}$ and $b = 0.4/\text{s}$, and x is the dependent variable that depends on t time. The oscillator has the initial conditions $x(0) = 1\text{m}$ and $\dot{x}(0) = \frac{1}{2}\text{m/s}$.

With these initial conditions, use your computer to plot $x(t)$ and $\dot{x}(t)$ verses t (time) for $t = 0$ to 10 seconds. Also make a phase plot of $\dot{x}(t)$ verses $x(t)$ for $t = 0$ to 30 seconds.

You may use any 2-D plotter that you like, so long as you can manage to produce a paper copy for your solution to this problem. Gnuplot is a free 2-D plotter that works on GNU/Linux, MS Windows, Mac, and more. You can get gnuplot at http://gnuplot.sourceforge.net/. The instructor used gnuplot to make the solutions to this homework. Another possibility is to use Mathematica.

5 two first order linear differential equations

The undriven over-damped harmonic oscillator

$$\ddot{x} = -\omega_0^2 x - 2\beta\dot{x}, \tag{5.1}$$

where ω_0 and β are constant parameters, and $\omega_2 \equiv \sqrt{\beta^2 - \omega_0^2}$ is a real constant that is greater than zero, can be rewritten as the two first order linear differential equations

$$\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -2\beta \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{5.2}$$

where x and $p \equiv \dot{x}$ are the two time dependent variables of the system.

Solve 5.2 finding the two eigenvalues and eigenvectors for this system along the way. Use the form of solution: $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \equiv \vec{a} e^{\lambda t}$, where $\vec{a}$ is an eigenvector and λ is an eigenvalue. Normalizing the eigenvectors is not required. Compare your two eigenvectors with the two dotted lines in the phase plot (with $p \equiv v$) in figure 3-11 (page 115) in Thornton and Marion. You can get a copy of it in the file TandM_figure3.11.pdf where you got this file.