This homework looks long only because it reiterates some of the content in the text and from lecture. Extra credit problem parts have been introduced in order to reduce the size of the required homework. Do the non-extra credit problems first. The score value of the extra credit problems is not very significant.

1 linear resonance response curve

Consider a driven damped harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A\cos\omega t, \tag{1.1}$$

where β, ω_0, A and ω are constant parameters, and t is time. We can call A the driving amplitude, and ω the driving angular frequency.

As explained in the text the steady state solution for $x(t)$, which we refer to as $x_p(t)$, can be written as

$$x_p(t) = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}\cos\left(\omega t - \delta\right) \tag{1.2}$$

with

$$\delta = \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right). \tag{1.3}$$

We define the response amplitude, D, as the factor that multiples the cos function so

$$D = \frac{A}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}. \tag{1.4}$$

The steady state solution, $x_p(t)$ does not depend on initial conditions like the transient (complementary) solutions. $x_p(t)$ is the solution that is approached after a long time after the transient (complementary) solution, which depended on initial conditions, has died away in an asymptotic fashion as time increased.

We define Q, the quality factor as

$$Q \equiv \frac{\omega_R}{2\beta}, \tag{1.5}$$

where ω_R is the value of the driving angular frequency, ω, when there is a the maximum amplitude, D, of $x_p(t)$ (the steady state solution). So that the steady state amplitude of $x_p(t)$ is maximized at an ω value equal to ω_R.

Show that $\omega_R^2 = \omega_0^2 - 2\beta^2$.

1.0 solution

To simplify things we can define

$$X(\omega) \equiv \left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2 \tag{1.6}$$

D is a monotonically decreasing function of X for all positive values of X, so

$$\frac{\mathrm{d}D}{\mathrm{d}X} < 0. \tag{1.7}$$

Using the chain rule we have

$$\frac{\mathrm{d}D}{\mathrm{d}\omega} = \frac{\mathrm{d}D}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\omega} \tag{1.8}$$

and so if

$$\left.\frac{\mathrm{d}D}{\mathrm{d}\omega}\right|_{\omega=\omega_R} = 0 \tag{1.9}$$

than it must be the case that

$$\left.\frac{\mathrm{d}X}{\mathrm{d}\omega}\right|_{\omega=\omega_R} = 0 \tag{1.10}$$

which gives

$$2\omega_R 2\left(\omega_R^2 - \omega_0^2\right) + 8\omega_R\beta^2 = 0 \qquad \Rightarrow \qquad \omega_R\left(\omega_R^2 - \omega_0^2 + 2\beta^2\right) = 0 \tag{1.11}$$

which gives

$$\omega_R = 0 \tag{1.12}$$

or

$$\omega_R^2 = \omega_0^2 - 2\beta^2. \tag{1.13}$$

So the solution of interest is

$$\boxed{\omega_R^2 = \omega_0^2 - 2\beta^2}. \tag{1.14}$$

We can check that this is a maximum in D, or concave down, at $\omega = \omega_R$ by showing that

$$\left.\frac{\mathrm{d}^2D}{\mathrm{d}\omega^2}\right|_{\omega=\omega_R} < 0. \tag{1.15}$$

So

$$\begin{aligned}\left.\frac{\mathrm{d}^2D}{\mathrm{d}\omega^2}\right|_{\omega=\omega_R} &= \left.\left[\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\mathrm{d}D}{\mathrm{d}X}\frac{\mathrm{d}X}{\mathrm{d}\omega}\right)\right]\right|_{\omega=\omega_R} = \left.\left(\frac{\mathrm{d}^2D}{\mathrm{d}\omega\mathrm{d}D}\frac{\mathrm{d}X}{\mathrm{d}\omega} + \frac{\mathrm{d}^2X}{\mathrm{d}\omega^2}\frac{\mathrm{d}D}{\mathrm{d}X}\right)\right|_{\omega=\omega_R} \\ &= \left.\frac{\mathrm{d}^2D}{\mathrm{d}\omega\mathrm{d}X}\right|_{\omega=\omega_R} 0 + \left.\left[\left(\omega_0^2-\omega^2\right)^2 + 4\omega^2\beta^2\right]\right|_{\omega=\omega_R} \left.\frac{\mathrm{d}D}{\mathrm{d}X}\right|_{\omega=\omega_R} = 8(\omega_0^2 - 2\beta^2)\left.\frac{\mathrm{d}D}{\mathrm{d}X}\right|_{\omega=\omega_R}\end{aligned} \tag{1.16}$$

and because D is a monotonically decreasing function of X

$$\left.\frac{\mathrm{d}D}{\mathrm{d}X}\right|_{\omega=\omega_R} < 0 \tag{1.17}$$

so

$$\left.\frac{\mathrm{d}^2D}{\mathrm{d}\omega^2}\right|_{\omega=\omega_R} < 0 \tag{1.18}$$

if long as

$$\omega_0^2 - 2\beta^2 > 0 \tag{1.19}$$

and then D is concave down at $\omega = \omega_R$.

2 scaling the driven simple harmonic oscillator

A sinusoidally driven damped simple harmonic oscillator can be modeled by

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\cos\omega t, \tag{2.1}$$

where m is mass, k is a linear spring constant, b is a linear damping constant, F_0 is the force amplitude of the driving force, and ω is the driving force angular frequency. So there are five physical parameters, but we know that we do not need to study a five-dimensional parameter space in order to study this system. We know that we can divide this equation by m, giving the following equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t, \tag{2.2}$$

which we can rewrite as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = A\cos\omega t, \tag{2.3}$$

where $\omega_0^2 \equiv \frac{k}{m}$, $2\beta \equiv \frac{b}{m}$, and $A \equiv \frac{F_0}{m}$ are derived parameters. We have reduced a five-dimensional parameter space to a four-dimensional parameter space, by introducing new derived parameters. A distinction between the system in equation 2.1 and that in the scaled equation 2.3, is that the scaled system in equation 2.3 cannot represent the unscaled system in equation 2.1 when we have zero mass, $m = 0$, but otherwise the results gotten from equation 2.3 can be easily transformed to our original "real" physical model in equation 2.1.

2.1 fully scaled

We continue the scaling process started here by introducing the change of variables $\eta = \frac{x}{x_0}$, and $\tau = \frac{t}{t_0}$, where x_0 and t_0 are yet to be determined functions of the four remaining parameters. Note that, if x_0 has units of length and t_0 has units of time, then η and τ will be dimensionless. Find x_0 and t_0 such that there are only two parameters left in this scaled differential equation for the new variable η in terms of derivatives with respect to the new independent variable τ, like so

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\nu\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \eta = \cos\Omega\tau, \tag{2.4}$$

and what are the two remaining derived parameters ν and Ω in terms of the five original physical parameters.

2.1 solution

Eliminating x and t from equation 2.3 in favor of η and τ gives

$$\frac{x_0}{t_0^2}\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\beta\frac{x_0}{t_0}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \omega_0^2 x_0\eta = A\cos(\omega t_0\tau) \;\Rightarrow\; \frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\beta t_0\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \omega_0^2 t_0^2\eta = \frac{t_0^2}{x_0}A\cos(\omega t_0\tau) \tag{2.5}$$

where we have used $t = t_0\tau$, $x = x_0\eta$, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{t_0}\frac{\mathrm{d}}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{1}{t_0^2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}$ which comes from the definitions given for variables η and τ. Comparing equation 2.4 with the right most equation in 2.5 we get

$$2\beta t_0 = 2\nu \tag{2.6}$$
$$\omega_0^2 t_0^2 = 1 \tag{2.7}$$
$$\frac{t_0^2}{x_0}A = 1 \tag{2.8}$$
$$\omega t_0 = \Omega. \tag{2.9}$$

Solving equations 2.8 through 2.9 gives

$$x_0 = \frac{A}{\omega_0^2} \tag{2.10}$$
$$t_0 = \frac{1}{\omega_0} \tag{2.11}$$
$$\nu = \frac{\beta}{\omega} \tag{2.12}$$
$$\Omega = \frac{\omega}{\omega_0}. \tag{2.13}$$

Writing this in terms of the original parameters gives the following length scale, x_0, time scale, τ, and two dimensionless parameters ν and Ω as

$$\boxed{x_0 = \frac{F_0}{k} \qquad t_0 = \sqrt{\frac{m}{k}} \qquad \nu = \frac{b}{2m\omega} \qquad \Omega = \omega\sqrt{\frac{m}{k}}}. \tag{2.14}$$

2.2 other scalings(5 pts extra credit)

The scaling of ODE models is not always unique. For each of the ordinary differential equations (ODEs) listed below, show whether or not our driven damped simple harmonic oscillator, from equation 2.1, may be scaled (linearly transformed) into that ODE.

$$\mu\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \eta = \cos\Omega\tau \tag{2.15}$$

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \eta = \Lambda\cos\Omega\tau \tag{2.16}$$

2.2 solution

For

$$\mu\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \eta = \cos\Omega\tau, \tag{2.17}$$

we can use the left most equation part of 2.5 divided by A to show that for this to be scaled version of equation 2.1 we must have the following equations satisfied

$$\mu = \frac{x_0}{t_0^2 A} \qquad 1 = \beta\frac{x_0}{t_0 A} \qquad 1 = \frac{\omega_0^2 x_0}{A} \qquad \Omega = \omega t_0. \tag{2.18}$$

Since there is nothing inconsistent about these equations, than equation 2.15 can be a scaled version of our driven damped simple harmonic oscillator. Of course it will not work in the cases where $m = 0$ and/or $A = 0$.

For

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \eta = \Lambda\cos\Omega\tau, \tag{2.19}$$

we can use the left most equation part of 2.5 to show that for this to be scaled version of equation 2.1 we must have the following equations satisfied

$$1 = \beta t_0 \qquad 1 = \omega_0^2 t_0^2 \qquad \Lambda = \frac{t_0^2}{x_0}A \qquad \Omega = \omega t_0. \tag{2.20}$$

Since these equation are not consistent unless we restrict values of parameters where $\beta = \omega_0$, this does not represent a the same form of the of equation of motion that equation 2.16 represents.

In summary: equation 2.15 **Yes**, and equation 2.16 **No**.

2.3 subjective question (5 pts extra credit)

What good are scaled ODE models? Please be brief.

2.3 solution

Scaling a physical model reduces the complexity of the model. After scaling there's less there to study. *Any reasonable answer will due here.*