

This homework looks long only because it reiterates some of the content in the text and from lecture. Extra credit problem parts have been introduced in order to reduce the size of the required homework. Do the non-extra credit problems first. The score value of the extra credit problems is not very significant.

## 1 linear resonance response curve

Consider a driven damped harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = A \cos \omega t, \quad (1.1)$$

where  $\beta$ ,  $\omega_0$ ,  $A$  and  $\omega$  are constant parameters, and  $t$  is time. We can call  $A$  the driving amplitude, and  $\omega$  the driving angular frequency.

As explained in the text the steady state solution for  $x(t)$ , which we refer to as  $x_p(t)$ , can be written as

$$x_p(t) = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}} \cos(\omega t - \delta) \quad (1.2)$$

with

$$\delta = \tan^{-1} \left( \frac{2\omega\beta}{\omega_0^2 - \omega^2} \right). \quad (1.3)$$

We define the response amplitude,  $D$ , as the factor that multiplies the cos function so

$$D = \frac{A}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\omega^2\beta^2}}. \quad (1.4)$$

The steady state solution,  $x_p(t)$  does not depend on initial conditions like the transient (complementary) solutions.  $x_p(t)$  is the solution that is approached after a long time after the transient (complementary) solution, which depended on initial conditions, has died away in an asymptotic fashion as time increased.

We define  $Q$ , the quality factor as

$$Q \equiv \frac{\omega_R}{2\beta}, \quad (1.5)$$

where  $\omega_R$  is the value of the driving angular frequency,  $\omega$ , when there is a the maximum amplitude,  $D$ , of  $x_p(t)$  (the steady state solution). So that the steady state amplitude of  $x_p(t)$  is maximized at an  $\omega$  value equal to  $\omega_R$ .

Show that  $\omega_R^2 = \omega_0^2 - 2\beta^2$ .

## 2 scaling the driven simple harmonic oscillator

A sinusoidally driven damped simple harmonic oscillator can be modeled by

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = F_0 \cos \omega t, \quad (2.1)$$

where  $m$  is mass,  $k$  is a linear spring constant,  $b$  is a linear damping constant,  $F_0$  is the force amplitude of the driving force, and  $\omega$  is the driving force angular frequency. So there are five physical parameters, but we know that we do not need to study a five-dimensional parameter space in order to study this system. We know that we can divide this equation by  $m$ , giving the following equation

$$\frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = \frac{F_0}{m} \cos \omega t, \quad (2.2)$$

which we can rewrite as

$$\frac{d^2x}{dt^2} + 2\beta \frac{dx}{dt} + \omega_0^2 x = A \cos \omega t, \quad (2.3)$$

where  $\omega_0^2 \equiv \frac{k}{m}$ ,  $2\beta \equiv \frac{b}{m}$ , and  $A \equiv \frac{F_0}{m}$  are derived parameters. We have reduced a five-dimensional parameter space to a four-dimensional parameter space, by introducing new derived parameters. A distinction between the system in equation 2.1 and that in the scaled equation 2.3, is that the scaled system in equation 2.3 cannot represent the unscaled system in equation 2.1 when we have zero mass,  $m = 0$ , but otherwise the results gotten from equation 2.3 can be easily transformed to our original “real” physical model in equation 2.1.

## 2.1 fully scaled

We continue the scaling process started here by introducing the change of variables  $\eta = \frac{x}{x_0}$ , and  $\tau = \frac{t}{t_0}$ , where  $x_0$  and  $t_0$  are yet to be determined functions of the four remaining parameters. Note that, if  $x_0$  has units of length and  $t_0$  has units of time, then  $\eta$  and  $\tau$  will be dimensionless. Find  $x_0$  and  $t_0$  such that there are only two parameters left in this scaled differential equation for the new variable  $\eta$  in terms of derivatives with respect to the new independent variable  $\tau$ , like so

$$\frac{d^2\eta}{d\tau^2} + 2\nu \frac{d\eta}{d\tau} + \eta = \cos \Omega\tau, \quad (2.4)$$

and what are the two remaining derived parameters  $\nu$  and  $\Omega$  in terms of the five original physical parameters.

## 2.2 other scalings(5 pts extra credit)

The scaling of ODE models is not always unique. For each of the ordinary differential equations (ODEs) listed below, show whether or not our driven damped simple harmonic oscillator, from equation 2.1, may be scaled (linearly transformed) into that ODE.

$$\mu \frac{d^2\eta}{d\tau^2} + 2 \frac{d\eta}{d\tau} + \eta = \cos \Omega\tau \quad (2.5)$$

$$\frac{d^2\eta}{d\tau^2} + 2 \frac{d\eta}{d\tau} + \eta = \Lambda \cos \Omega\tau \quad (2.6)$$

## 2.3 subjective question (5 pts extra credit)

What good are scaled ODE models? Please be brief.