1 The Gradient Operator, ∇, in Spherical Coordinates

The ∇ operator can be defined using

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}, \tag{1.1}$$

where $\mathrm{d}f$ is the differential of an arbitrary scaler function of position in three dimensional space, and $\mathrm{d}\vec{r}$ is the differential of a displacement vector $\vec{r}$ in this three dimensional space, which in spherical coordinates can be written as

$$\mathrm{d}\vec{r} = \hat{r}\,\mathrm{d}r + \hat{\theta}\,r\,\mathrm{d}\theta + \hat{\phi}\,r\sin\theta\,\mathrm{d}\phi. \tag{1.2}$$

Show that the ∇ operator when expressed in spherical coordinates, $(r,\ \theta,\ \phi)$, can be written as

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{1.3}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are the corresponding mutually perpendicular spherical coordinate unit vectors. Hint: Express $\mathrm{d}f$ in terms of partial derivatives with respect to r, θ, and ϕ, and use $\mathrm{d}\vec{r}$ from equation 1.2 in equation 1.1.

1.0 solution

$$\mathrm{d}f(r,\theta,\phi) = \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\mathrm{d}\phi \tag{1.4}$$

and with 1.1 and 1.2

$$\frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial \theta}\mathrm{d}\theta + \frac{\partial f}{\partial \phi}\mathrm{d}\phi = \nabla f \cdot \left(\hat{r}\,\mathrm{d}r + \hat{\theta}\,r\,\mathrm{d}\theta + \hat{\phi}\,r\sin\theta\,\mathrm{d}\phi\right) \tag{1.5}$$

$$\Rightarrow\quad \left(\frac{\partial f}{\partial r} - \nabla f \cdot \hat{r}\right)\mathrm{d}r + \left(\frac{\partial f}{\partial \theta} - r\,\nabla f \cdot \hat{\theta}\right)\mathrm{d}\theta + \left(\frac{\partial f}{\partial \phi} - r\sin\theta\,\nabla f \cdot \hat{\phi}\right)\mathrm{d}\phi = 0 \tag{1.6}$$

Since $\mathrm{d}r$, $\mathrm{d}\theta$, and $\mathrm{d}\phi$ are independent, the coefficients of $\mathrm{d}r$, $\mathrm{d}\theta$, and $\mathrm{d}\phi$ must each be zero. So

$$\begin{array}{llll} & \frac{\partial f}{\partial r} - \nabla f \cdot \hat{r} = 0 & \frac{\partial f}{\partial \theta} - r\,\nabla f \cdot \hat{\theta} = 0 & \frac{\partial f}{\partial \phi} - r\sin\theta\,\nabla f \cdot \hat{\phi} = 0 \\ \Rightarrow & \nabla f \cdot \hat{r} = \frac{\partial f}{\partial r} & \nabla f \cdot \hat{\theta} = \frac{1}{r}\frac{\partial f}{\partial \theta} & \nabla f \cdot \hat{\phi} = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \end{array} \tag{1.7}$$

So we see the components of ∇f in the $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ directions. The unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are mutually perpendicular, so that the projections of ∇f onto them are the corresponding components of ∇f. Writing ∇f as a vector with the three mutually perpendicular components gives

$$\nabla f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}. \tag{1.8}$$

so that

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \tag{1.9}$$

2 Varying Radial Mass Density Causes Uniform Radial $\vec{g}$ Field

There is a radially distributed mass distribution, $\rho(r)$. The gravitational vector field, from this mass distribution, does not depend on the radial position r, and is equal to $\vec{g} = -C\,\hat{r}$, where C is a positive constant, and $\hat{r}$ is the radial unit vector. Find $\rho(r)$. Hint: $-\nabla\Phi = \vec{g}$ and $\nabla^2\Phi = 4\pi G\,\rho(r)$.

2.0 solution

We can put together

$$-\nabla\Phi = \vec{g} \quad , \quad \nabla^2\Phi = 4\pi G\,\rho(r) \quad \text{and} \quad \vec{g} = -C\,\hat{r} \tag{2.1}$$

to give

$$\nabla\cdot(\nabla\Phi) = 4\pi G\,\rho(r) \quad \Rightarrow \quad \nabla\cdot(C\,\hat{r}) = 4\pi G\,\rho(r) \quad \Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 C\right) = 4\pi G\,\rho(r) \tag{2.2}$$

which gives

$$\frac{1}{r^2}(2rC) = 4\pi G\,\rho(r) \quad \Rightarrow \quad \boxed{\rho(r) = \frac{C}{2\pi G r}} \tag{2.3}$$

3 Simple Harmonic Motion using Gravity

A hole goes straight through the center of the earth, from one side of the earth to the opposing side of the earth. A ball is dropped from the surface of the earth down the hole. How long will it take the ball to come back to the surface of the earth where the ball was dropped from? You can make the approximating assumptions that the earth is a uniform sphere with radius $R_E = 6.38\times10^6$ m, the acceleration due to gravity on the surface is $g = 9.8\frac{\text{m}}{\text{s}^2}$, the earth is not moving, there is no air friction, and the diameter of the hole is very small compared to the radius of the earth. You may use results from examples in your text. Remember that $g = G\frac{M_E}{R_E^2}$.

3.0 solution

The force on the ball with mass m at a distance r in from the center of the earth (sphere) is equal to the force from just the mass of the earth that is contained within a sphere of radius r. The mass in a sphere of radius r, that is inside of sphere of radius R_E is

$$M(r) = M_E\frac{r^3}{R_E^3}. \tag{3.1}$$

Newton's second law applied to the sphere gives

$$m\ddot{r} = -G\frac{mM(r)}{r^2} \quad \Rightarrow \quad m\ddot{r} = -G\frac{m\left(M_E\frac{r^3}{R_E^3}\right)}{r^2} \quad \Rightarrow \quad \ddot{r} = -G\frac{M_E}{R_E^2}\frac{r}{R_E} \quad \Rightarrow \quad \ddot{r} = -\frac{g}{R_E}r. \tag{3.2}$$

So the equation of motion is that of simple harmonic motion with angular frequency $\omega_0 = \sqrt{\frac{g}{R_E}}$. So the period of the motion, which is the time to return back to the starting position is

$$\tau = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{R_E}{g}} = 2\pi\sqrt{\frac{6.38\times10^6\,\text{m}}{9.8\frac{\text{m}}{\text{s}^2}}} \approx 5069.64\text{s} \approx \boxed{84.5\,\text{minutes}}. \tag{3.3}$$

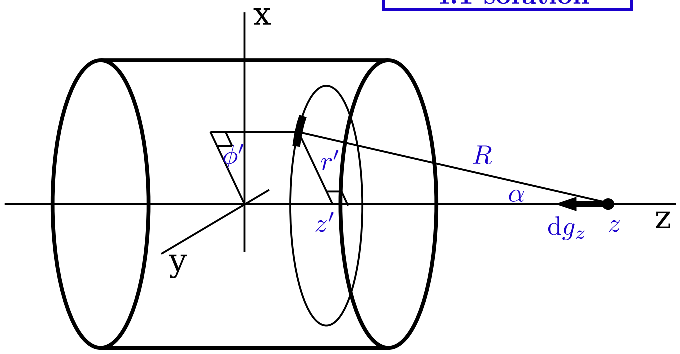
4 Calculate $\vec{g}$ Field from a Cylinder

There exists a solid cylinder with length L, radius a, and total mass M, that is uniformly distributed.

4.1 Without Potential

Calculate the gravitational field vector, $\vec{g}$, due to this cylinder for positions outside the cylinder along the axis of the cylinder. Do this without calculating gravitational potential. Use cylindrical coordinates (r, ϕ, z) with the cylinder centered at the origin, and with the z-axis lined up along the axis of the cylinder.

4.1 solution



Let z be along the axis of the cylinder. The volume of a small piece of ring, in the above figure, $\mathrm{d}V$, is

$$\mathrm{d}V = r'\mathrm{d}\phi'\,\mathrm{d}z'\,\mathrm{d}r' \tag{4.1}$$

where we have introduced a prime ($'$) for the variables that we will sum over. From symmetry the net force will be in the $-z$ direction, as shown in the figure above. The z component of the gravitational field from the piece of ring shown in the figure is

$$\mathrm{d}g_z = -G\frac{\mathrm{d}m}{R^2}\cos\alpha = -G\frac{\rho\mathrm{d}V}{R^2}\cos\alpha = -G\frac{\rho\, r'\mathrm{d}\phi'\,\mathrm{d}z'\mathrm{d}r'}{R^2}\cos\alpha = -Gr'\mathrm{d}\phi'\,\mathrm{d}z'\,\mathrm{d}r'\frac{\cos\alpha}{R^2} \tag{4.2}$$

where we have in introduce ρ as the mass density of the cylinder, and we have included the minus sign to show that this field is in the $-z$ direction. We see that there is a right triangle with the angle α such that the adjacent is $z - z'$ and the hypotenuse is

$$R = \sqrt{(z - z')^2 + r'^2} \tag{4.3}$$

and

$$\cos\alpha = \frac{z - z'}{\sqrt{(z - z')^2 + r'^2}} \tag{4.4}$$

where z is the location of the point where we are computing the gravitational field, and z' and r' are position variables inside the cylinder that will be summed over. Plugging these results into equation 4.2 gives

$$\mathrm{d}g_z = -G\rho\, r'\mathrm{d}z'\,\mathrm{d}r'\,\mathrm{d}\phi'\frac{z - z'}{\left[(z - z')^2 + r'^2\right]^{\frac{3}{2}}}. \tag{4.5}$$

We can sum over the ring, whereby removing the $\mathrm{d}\phi'$ giving

$$\mathrm{d}g_z = -2\pi G\rho\, r'\mathrm{d}z'\,\mathrm{d}r'\frac{z - z'}{\left[(z - z')^2 + r'^2\right]^{\frac{3}{2}}}. \tag{4.6}$$

We replace ρ with it's M, L, and a dependence giving

$$\mathrm{d}g_z = -2\pi G\left(\frac{M}{\pi a^2 L}\right) r'\mathrm{d}z'\,\mathrm{d}r' \frac{z-z'}{\left[(z-z')^2+r'^2\right]^{\frac{3}{2}}} = -2G\frac{M}{a^2L} r'\mathrm{d}z'\,\mathrm{d}r' \frac{z-z'}{\left[(z-z')^2+r'^2\right]^{\frac{3}{2}}}. \tag{4.7}$$

We sum over the whole cylinder giving

$$\int_{g_z=0}^{g} \mathrm{d}g_z = g_z = \int_{z'=-\frac{L}{2}}^{\frac{L}{2}} \int_{r'=0}^{a} \left\{ -2G\frac{M}{a^2L} r'\mathrm{d}z'\,\mathrm{d}r' \frac{z-z'}{\left[(z-z')^2+r'^2\right]^{\frac{3}{2}}} \right\} \tag{4.8}$$

$$= -2G\frac{M}{a^2L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}} (z-z')\,\mathrm{d}z' \int_{r'=0}^{a} \frac{r'\mathrm{d}r'}{\left[r'^2+(z-z')^2\right]^{\frac{3}{2}}}. \tag{4.9}$$

From integral tables we find

$$\int \frac{x\,\mathrm{d}x}{(x^2+b^2)^{\frac{3}{2}}} = -\frac{1}{\sqrt{x^2+b^2}}. \tag{4.10}$$

So with $x = r$, $\mathrm{d}x = \mathrm{d}r$, and $b = z - z'$ we have

$$g_z = -2G\frac{M}{a^2L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}} (z-z')\,\mathrm{d}z' \left[-\frac{1}{\sqrt{r'^2+(z-z')^2}}\right]\Bigg|_{r'=0}^{a} \tag{4.11}$$

$$= -2G\frac{M}{a^2L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}} (z-z')\,\mathrm{d}z' \left(\frac{1}{z-z'} - \frac{1}{\sqrt{a^2+(z-z')^2}}\right) \tag{4.12}$$

$$= -2G\frac{M}{a^2L}\left(\int_{z'=-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}z' - \int_{z'=-\frac{L}{2}}^{\frac{L}{2}} \frac{(z-z')\,\mathrm{d}z'}{\sqrt{a^2+(z-z')^2}}\right) \tag{4.13}$$

$$= -2G\frac{M}{a^2L}\left(L - \int_{z'=-\frac{L}{2}}^{\frac{L}{2}} \frac{(z-z')\,\mathrm{d}z'}{\sqrt{(z-z')^2+a^2}}\right). \tag{4.14}$$

We can use, integral tables,

$$\int \frac{x\,\mathrm{d}x}{\sqrt{x^2+b^2}} = \sqrt{x^2+b^2} \tag{4.15}$$

with $x = z - z'$, $\mathrm{d}x = -\mathrm{d}z'$, and $b = a$, so

$$g_z = -2G\frac{M}{a^2L}\left(L + \sqrt{(z-z')^2+a^2}\,\Bigg|_{z'=-\frac{L}{2}}^{\frac{L}{2}}\right) \tag{4.16}$$

$$= \boxed{-2G\frac{M}{a^2L}\left(L + \sqrt{\left(z-\frac{L}{2}\right)^2+a^2} - \sqrt{\left(z+\frac{L}{2}\right)^2+a^2}\right)}. \tag{4.17}$$

It's good to note, but not obvious that this result, that g should have a $\frac{1}{z^2}$ dependence for large z values, like the case for a point mass. Put another way, $g_z \propto \frac{1}{z^2}$ for $z \gg a$ and $z \gg L$.

4.2 Compute Potential First

Calculate the gravitational field vector, $\vec{g}$, due to this cylinder for positions outside the cylinder along the axis of the cylinder. In this case calculate the gravitational potential first, and use that to get the gravitational field vector. As in subsection 4.1, use cylindrical coordinates (r, ϕ, z) with the cylinder centered at the origin, and with the z-axis lined up along the axis of the cylinder.

4.2 solution

In a simular fashion we start with the gradational potential $\mathrm{d}\Phi$ for a small source mass $\mathrm{d}m$

$$\mathrm{d}\Phi = -G\frac{\mathrm{d}m}{R} = -G\frac{\rho \mathrm{d}V}{R} = -G\frac{\rho\, r'\mathrm{d}\phi'\,\mathrm{d}z'\mathrm{d}r'}{R} = -G\frac{r'\mathrm{d}\phi'\,\mathrm{d}z'\,\mathrm{d}r'}{R}. \tag{4.18}$$

As before

$$R = \sqrt{(z-z')^2 + r'^2}, \tag{4.19}$$

so we have

$$\mathrm{d}\Phi = -G\rho\frac{r'\mathrm{d}\phi'\,\mathrm{d}z'\,\mathrm{d}r'}{\sqrt{(z-z')^2 + r'^2}}. \tag{4.20}$$

We can sum over the ring, whereby removing the $\mathrm{d}\phi'$ giving

$$\mathrm{d}\Phi = -2\pi G\rho\frac{r'\mathrm{d}z'\,\mathrm{d}r'}{\sqrt{(z-z')^2 + r'^2}} \tag{4.21}$$

and replacing ρ with it's M, L, and a dependence gives

$$\mathrm{d}\Phi = -2\pi G\left(\frac{M}{\pi a^2 L}\right)\frac{r'\mathrm{d}z'\,\mathrm{d}r'}{\sqrt{(z-z')^2 + r'^2}} = -2G\frac{M}{a^2 L}\frac{r'\mathrm{d}z'\,\mathrm{d}r'}{\sqrt{(z-z')^2 + r'^2}}. \tag{4.22}$$

We sum over the whole cylinder giving

$$\Phi = -2G\frac{M}{a^2 L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\int_{r'=0}^{a}\frac{r'\mathrm{d}z'\,\mathrm{d}r'}{\sqrt{(z-z')^2 + r'^2}} \tag{4.23}$$

$$= -2G\frac{M}{a^2 L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}z'\int_{r'=0}^{a}\frac{r'\,\mathrm{d}r'}{\sqrt{r'^2 + (z-z')^2}}. \tag{4.24}$$

From integral tables we can use

$$\int\frac{x\,\mathrm{d}x}{\sqrt{x^2+b^2}} = \sqrt{x^2+b^2} \tag{4.25}$$

where $x = r'$, $\mathrm{d}x = \mathrm{d}r'$, and $b = z - z'$ giving

$$\Phi = -2G\frac{M}{a^2 L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}z'\left[\sqrt{r'^2 + (z-z')^2}\Big|_{r'=0}^{a}\right] \tag{4.26}$$

$$= -2G\frac{M}{a^2 L}\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\mathrm{d}z'\left[\sqrt{a^2 + (z-z')^2} - (z-z')\right] \tag{4.27}$$

$$= -2G\frac{M}{a^2 L}\left[\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\sqrt{a^2 + (z-z')^2}\mathrm{d}z' - \int_{z'=-\frac{L}{2}}^{\frac{L}{2}}(z-z')\,\mathrm{d}z'\right] \tag{4.28}$$

$$= -2G\frac{M}{a^2 L}\left\{\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\sqrt{a^2 + (z-z')^2}\mathrm{d}z' + \left[\frac{1}{2}(z'-z)^2\Big|_{z'=-\frac{L}{2}}^{\frac{L}{2}}\right]\right\} \tag{4.29}$$

$$= -2G\frac{M}{a^2 L}\left[\int_{z'=-\frac{L}{2}}^{\frac{L}{2}}\sqrt{a^2 + (z'-z)^2}\mathrm{d}z' + \frac{1}{2}\left(\frac{L}{2} - z\right)^2 - \frac{1}{2}\left(-\frac{L}{2} - z\right)^2\right]. \tag{4.30}$$

From integral tables we can use

$$\int \sqrt{x^2+b^2}\,\mathrm{d}x = \frac{1}{2}\left[x\,\sqrt{x^2+b^2} + b^2\,\ln\left(x+\sqrt{x^2+b^2}\right)\right] \tag{4.31}$$

where $x = z' - z$, $\mathrm{d}x = \mathrm{d}z'$, and $b = a$ giving

$$\Phi(z) = -2G\frac{M}{a^2L}\left\{\frac{1}{2}\left[(z'-z)\,\sqrt{(z'-z)^2+a^2} + a^2\,\ln\left((z'-z)+\sqrt{(z'-z)^2+a^2}\right)\right]\Bigg|_{z'=-\frac{L}{2}}^{\frac{L}{2}} - Lz\right\} \tag{4.32}$$

$$\tag{4.33}$$

$$= -G\frac{M}{a^2L}\left\{\left[\left(\frac{L}{2}-z\right)\sqrt{\left(\frac{L}{2}-z\right)^2+a^2} + a^2\,\ln\left(\left(\frac{L}{2}-z\right)+\sqrt{\left(\frac{L}{2}-z\right)^2+a^2}\right)\right]\right.$$
$$\left. - \left[\left(-\frac{L}{2}-z\right)\sqrt{\left(-\frac{L}{2}-z\right)^2+a^2} + a^2\,\ln\left(\left(-\frac{L}{2}-z\right)+\sqrt{\left(-\frac{L}{2}-z\right)^2+a^2}\right)\right] - 2Lz\right\}$$

$$\tag{4.34}$$

$$= G\frac{M}{a^2L}\left[\left(z-\frac{L}{2}\right)\sqrt{\left(z-\frac{L}{2}\right)^2+a^2} - a^2\,\ln\left(\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}-\left(z-\frac{L}{2}\right)\right)\right.$$
$$\left. - \left(z+\frac{L}{2}\right)\sqrt{\left(z+\frac{L}{2}\right)^2+a^2} + a^2\,\ln\left(\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\left(z+\frac{L}{2}\right)\right) + 2Lz\right]. \tag{4.35}$$

So

$$
\begin{aligned}
\frac{g_z}{G\frac{M}{a^2L}} &= -\frac{\frac{\partial\Phi}{\partial z}}{G\frac{M}{a^2L}} && (4.36)\\
&= \sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}+a^2\frac{\frac{z-\frac{L}{2}}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}+1}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\left(z-\frac{L}{2}\right)}\\
&\quad -\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}+a^2\frac{\frac{z+\frac{L}{2}}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}-1}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\left(z+\frac{L}{2}\right)}-2L && (4.37)\\
&= \sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}+a^2\frac{\frac{z-\frac{L}{2}+\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\left(z-\frac{L}{2}\right)}\\
&\quad -\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}+a^2\frac{\frac{z+\frac{L}{2}-\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\left(z+\frac{L}{2}\right)}-2L && (4.38)\\
&= \sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}+a^2\frac{1}{\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}}\\
&\quad -\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}-a^2\frac{1}{\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}}-2L && (4.39)\\
&= \sqrt{\left(z-\frac{L}{2}\right)^2+a^2}+\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}\\
&\quad -\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-2L && (4.40)\\
&= 2\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}-2\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}-2L, && (4.41)
\end{aligned}
$$

which gives

$$
\boxed{g_z=-2G\frac{M}{a^2L}\left(L+\sqrt{\left(z-\frac{L}{2}\right)^2+a^2}-\sqrt{\left(z+\frac{L}{2}\right)^2+a^2}\right)}, \qquad (4.42)
$$

as before.

5 Energy to Make a Planet

Calculate the energy needed to assemble a uniformly distributed spherical mass, with mass M and radius R, given that initially all the mass was completely dispersed (spread out far). This energy will be less then zero.

5.0 solution

We will pull the mass from far away into a shell of thickness $\mathrm{d}r$ onto the sphere, making the sphere thicker by $\mathrm{d}r$ each time. The energy difference between a thin shell that is spread out far, ∞, and the same mass that is in a shell that has a radius r and thickness $\mathrm{d}r$ around a preexisting mass of mass $M(r) = \frac{4}{3}\pi r^3 \rho$ is

$$\mathrm{d}U = -G\frac{M_{\text{shell}}\, M(r)}{r} - G\frac{\left(4\pi r^2 \mathrm{d}r \rho\right)\left(\frac{4}{3}\pi r^3 \rho\right)}{r} = -\frac{16}{3}\pi^2 G \rho^2 r^4 \mathrm{d}r. \tag{5.1}$$

where ρ is the final mass density of the sphere we are making. So the total energy change in moving all the mass M to a sphere of radius R is the integrated sum

$$\Delta U = \int_{U'=0}^{\Delta U} \mathrm{d}U' = -\int_{r'=0}^{R} \frac{16}{3} G\pi^2 \rho^2 r'^4 \, \mathrm{d}r' = -\frac{16}{3} G\pi^2 \rho^2 \frac{1}{5}\, r'^5 \Big|_{r'=0}^{R} = -\frac{16}{15} G\pi^2 \rho^2 R^5. \tag{5.2}$$

The density, ρ, can be written as a function of M and R so

$$\Delta U = -\frac{16}{15} G\pi^2 \left(\frac{M}{\frac{4}{3}\pi R^3}\right)^2 R^5 = \boxed{-\frac{3}{5} G \frac{M^2}{R}}. \tag{5.3}$$