1 The Gradient Operator, ∇, in Spherical Coordinates

The ∇ operator can be defined using

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\vec{r}, \tag{1.1}$$

where $\mathrm{d}f$ is the differential of an arbitrary scaler function of position in three dimensional space, and $\mathrm{d}\vec{r}$ is the differential of a displacement vector $\vec{r}$ in this three dimensional space, which in spherical coordinates can be written as

$$\mathrm{d}\vec{r} = \hat{r}\,\mathrm{d}r + \hat{\theta}\,r\,\mathrm{d}\theta + \hat{\phi}\,r\sin\theta\,\mathrm{d}\phi. \tag{1.2}$$

Show that the ∇ operator when expressed in spherical coordinates, (r, θ, ϕ), can be written as

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \tag{1.3}$$

where $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are the corresponding mutually perpendicular spherical coordinate unit vectors. Hint: Express $\mathrm{d}f$ in terms of partial derivatives with respect to r, θ, and ϕ, and use $\mathrm{d}\vec{r}$ from equation 1.2 in equation 1.1.

2 Varying Radial Mass Density Causes Uniform Radial $\vec{g}$ Field

There is a radially distributed mass distribution, $\rho(r)$. The gravitational vector field, from this mass distribution, does not depend on the radial position r, and is equal to $\vec{g} = -C\,\hat{r}$, where C is a positive constant, and $\hat{r}$ is the radial unit vector. Find $\rho(r)$. Hint: $-\nabla\Phi = \vec{g}$ and $\nabla^2\Phi = 4\pi G\,\rho(r)$.

3 Simple Harmonic Motion using Gravity

A hole goes straight through the center of the earth, from one side of the earth to the opposing side of the earth. A ball is dropped from the surface of the earth down the hole. How long will it take the ball to come back to the surface of the earth where the ball was dropped from? You can make the approximating assumptions that the earth is a uniform sphere with radius $R_E = 6.38 \times 10^6\,\mathrm{m}$, the acceleration due to gravity on the surface is $g = 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$, the earth is not moving, there is no air friction, and the diameter of the hole is very small compared to the radius of the earth. You may use results from examples in your text. Remember that $g = G\frac{M_E}{R_E^2}$.

4 Calculate $\vec{g}$ Field from a Cylinder

There exists a solid cylinder with length L, radius a, and total mass M, that is uniformly distributed.

4.1 Without Potential

Calculate the gravitational field vector, $\vec{g}$, due to this cylinder for positions outside the cylinder along the axis of the cylinder. Do this without calculating gravitational potential. Use cylindrical coordinates (r, ϕ, z) with the cylinder centered at the origin, and with the z-axis lined up along the axis of the cylinder.

4.2 Compute Potential First

Calculate the gravitational field vector, $\vec{g}$, due to this cylinder for positions outside the cylinder along the axis of the cylinder. In this case calculate the gravitational potential first, and use that to get the gravitational field vector. As in subsection 4.1, use cylindrical coordinates (r, ϕ, z) with the cylinder centered at the origin, and with the z-axis lined up along the axis of the cylinder.

5 Energy to Make a Planet

Calculate the energy needed to assemble a uniformly distributed spherical mass, with mass M and radius R, given that initially all the mass was completely dispersed (spread out far). This energy will be less then zero.