1 Shooting the Moon

In this problem assume that there is no atmospheric friction, and use a static model for the earth and moon system. Use: the mass of the earth is $M_E = 5.97 \times 10^{24}$kg, the mass of the moon is $M_m = 7.35 \times 10^{22}$kg, the radius of the earth is $R_E = 6.38 \times 10^6$m, and radius of the moon is $R_m = 1.74 \times 10^6$, the orbit radius of the earth moon system as measured from earth's center to the moon's center is $R_o = 3.84 \times 10^8$m, and the gravitation constant $G = 6.67 \times 10^{-11}\text{Nm}^2/\text{kg}^2$.

Find the minimum initial speed of a projectile that is shot to the surface of the moon, v_i. You must keep at least three significant figures. (The escape speed for earth is about $11200\frac{\text{m}}{\text{s}}$, and this is not much different.)

1.0 solution

Let the mass of the projectile be m. This projectile must just overcome the gravitational potential energy barrier that is between the earth and the moon and then the projectile will fall toward the moon. At a position between the earth and the moon there will be no net gravitational force on the projectile. Let this position measured relative to the center of the earth be R_b. Balancing both of the gravitational forces on the projectile gives

$$G\frac{mM_E}{R_b^2} = G\frac{mM_m}{(R_o - R_b)^2} \quad \Rightarrow \quad M_E(R_o - R_b)^2 - M_m R_b^2 = 0 \quad \Rightarrow \quad \left(1 - \frac{M_m}{M_E}\right)R_b^2 - 2R_oR_b + R_o^2 = 0. \tag{1.1}$$

Solving for R_b gives

$$R_b = \frac{2R_o \pm \sqrt{4R_o^2 - 4\left(1 - \frac{M_m}{M_E}\right)R_o^2}}{2\left(1 - \frac{M_m}{M_E}\right)} \quad \Rightarrow \quad R_b = R_o\frac{\left(1 \pm \sqrt{\frac{M_m}{M_E}}\right)}{1 - \frac{M_m}{M_E}} \tag{1.2}$$

The solution that is between the earth and the moon, where the forces oppose each other, is

$$R_b = R_o\frac{\left(1 - \sqrt{\frac{M_m}{M_E}}\right)}{1 - \frac{M_m}{M_E}}. \tag{1.3}$$

Using conservation of energy where the initial energy is the energy of the projectile when it just leaves the surface of the earth E_E and the final energy is when the projectile is at R_b, E_b, gives

$$E_E = E_b \quad \Rightarrow \quad -G\frac{mM_E}{R_E} - G\frac{mM_m}{R_o - R_E} + \frac{1}{2}mv_i^2 = -G\frac{mM_E}{R_b} - G\frac{mM_m}{R_o - R_b} + 0 \tag{1.4}$$

which gives

$$v_i^2 = +2G\frac{M_E}{R_E} + 2G\frac{M_m}{R_o - R_E} - 2G\frac{M_E}{R_b} - 2G\frac{M_m}{R_o - R_b} = +2G\left(\frac{M_E}{R_E} + \frac{M_m}{R_o - R_E} - \frac{M_E}{R_b} - \frac{M_m}{R_o - R_b}\right) \tag{1.5}$$

which gives

$$v_i = \sqrt{2G\left(\frac{M_E}{R_E} + \frac{M_m}{R_o - R_E} - \frac{M_E}{R_b} - \frac{M_m}{R_o - R_b}\right)}. \tag{1.6}$$

Plugging in numbers gives

$$\frac{M_m}{M_E} = \frac{7.35 \times 10^{22}}{5.97 \times 10^{24}} \approx 0.0123116 \tag{1.7}$$

$$R_b = R_o\frac{\left(1 - \sqrt{\frac{M_m}{M_E}}\right)}{1 - \frac{M_m}{M_E}} \approx 3.4564 \times 10^8\text{m} \tag{1.8}$$

$$v_i \approx \sqrt{2 \times 6.67 \times 10^{-11}\frac{\text{Nm}^2}{\text{kg}^2}\left(\frac{5.97 \times 10^{24}\text{kg}}{6.38 \times 10^6\text{m}} + \frac{7.35 \times 10^{22}\text{kg}}{3.84 \times 10^8\text{m} - 6.38 \times 10^6\text{m}} - \frac{5.97 \times 10^{24}\text{kg}}{3.4564 \times 10^8\text{m}} - \frac{7.35 \times 10^{22}\text{kg}}{3.84 \times 10^8\text{m} - 3.4564 \times 10^8\text{m}}\right)}$$
$$\approx \boxed{11064\frac{\text{m}}{\text{s}}}. \tag{1.9}$$