1 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \left(\frac{1}{2}\, {y'}^2 + a\, y \right) \mathrm{d}x, \tag{1.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$, and a is a constant.

Hints: Do so by using the Euler equation,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{1.2}$$

where $f(y, y'; x) = \frac{1}{2}\, {y'}^2 + a\, y$. You do not have to determine the two constants of integration, just call them c_1 and c_2.

1.0 solution

For J to be stationary Euler's equation must be satisfied, so

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{1.3}$$

where

$$f(y, y'; x) = \frac{1}{2}\, {y'}^2 + ay. \tag{1.4}$$

This gives

$$a - \frac{\mathrm{d}}{\mathrm{d}x}(y') = 0 \quad \Rightarrow \quad y'' = a \quad \Rightarrow \quad \int \frac{\mathrm{d}y'}{\mathrm{d}x}\,\mathrm{d}x = \int a\,\mathrm{d}x \quad \Rightarrow \quad y' = ax + c_1 \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int (ax + c_1)\,\mathrm{d}x \tag{1.5}$$

$$\Rightarrow \quad \boxed{y(x) = \frac{1}{2}ax^2 + c_1 x + c_2}. \tag{1.6}$$

2 Shortest Line Between Two Points

Show that the shortest line between two points in a plane is a straight line.

2.0 solution

We wish to minimize the path distance between two points (x_1, y_1) and (x_2, y_2). This path distance can be expressed as

$$J = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}. \tag{2.1}$$

Since y is just a function of x

$$\mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = y'\,\mathrm{d}x, \tag{2.2}$$

where we have defined

$$\frac{\mathrm{d}y}{\mathrm{d}x} \equiv y'. \tag{2.3}$$

So we may write J as

$$J = \int_{x_1}^{x_2} \sqrt{\mathrm{d}x^2 + (y'\,\mathrm{d}x)^2} = \int_{x_1}^{x_2} \sqrt{\left(1+y'^2\right)\mathrm{d}x^2} = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,\mathrm{d}x. \tag{2.4}$$

For J to be a minimum it is necessary that Euler's equation be satisfied, so

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0, \tag{2.5}$$

where

$$f = \sqrt{1+y'^2}. \tag{2.6}$$

Plugging f into 2.5 gives

$$\begin{aligned} & 0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) \quad \Rightarrow \quad \frac{y'}{\sqrt{1+y'^2}} = c_1 \quad \Rightarrow \quad y' = c_1\sqrt{1+y'^2} \quad \Rightarrow \quad y'^2 = c_1^2\left(1+y'^2\right) \\ & \Rightarrow \quad y'^2 = c_1^2 + c_1^2 y'^2 \quad \Rightarrow \quad y'^2\left(1-c_1^2\right) = c_1^2 \quad \Rightarrow \quad y' = \frac{c_1}{\sqrt{1-c_1^2}} \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int \frac{c_1}{\sqrt{1-c_1^2}}\,\mathrm{d}x \\ & \Rightarrow \quad y = \frac{c_1}{\sqrt{1-c_1^2}}\,x + c_2, \end{aligned} \tag{2.7}$$

which is the equation of a straight line, where c_1 and c_2 are constants of integration that may be adjusted to connect the two points, (x_1, y_1) and (x_2, y_2).

3 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x}\,\mathrm{d}x, \tag{3.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. What common geometry does $y(x)$ represent?

3.0 solution

We plug

$$f = \frac{\sqrt{1+y'^2}}{x} \tag{3.2}$$

into Euler's equation to get

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \quad \Rightarrow \quad \frac{1}{x}\frac{y'}{\sqrt{1+y'^2}} = c_1 \quad \Rightarrow \quad y' = c_1 x\sqrt{1+y'^2} \quad \Rightarrow \quad y'^2 = c_1^2 x^2\left(1+y'^2\right) \tag{3.3}$$

$$\Rightarrow \quad y'^2\left(1-c_1^2x^2\right) = c_1^2x^2 \quad \Rightarrow \quad y'^2 = \frac{c_1^2x^2}{1-c_1^2x^2} \quad \Rightarrow \quad y'\,\mathrm{d}x = \pm\frac{c_1x}{\sqrt{1-c_1^2x^2}}\,\mathrm{d}x \tag{3.4}$$

$$\Rightarrow \quad y = \int \pm\frac{c_1x}{\sqrt{1-c_1^2x^2}}\,\mathrm{d}x \quad \Rightarrow \quad y = \mp\frac{1}{c_1}\sqrt{1-c_1^2x^2} + c_2 \quad \Rightarrow \quad c_1y - c_1c_2 = \mp\sqrt{1-c_1^2x^2} \tag{3.5}$$

$$\Rightarrow \quad c_1^2 (y - c_2)^2 = 1 - c_1^2 x^2 \quad \Rightarrow \quad c_1^2 x^2 + c_1^2 (y - c_2)^2 = 1 \quad \Rightarrow \quad \boxed{x^2 + (y - c_2)^2 = \frac{1}{c_1^2}}. \tag{3.6}$$

This is the equation of a circle that is centered at $x = 0$ and who's y coordinate of the center position and radius depend on the positions (x_1, y_1) and (x_2, y_2).

4 Application of Fermat's Principle

Fermat's principle states that light traverses the path between two points which takes the least time.

The index of refraction for light traveling in a medium n is defined as $n \equiv \frac{c}{v}$, where c is the speed of light in a vacuum, and v is the speed of light in the medium.

Find the path that light will travel, $y(x)$, in a medium with an index of refraction $n(x)$ that is proportional to $1 + a\,x$, where a is a constant. The light travels in a region of space where x is positive. There is no variation of n along the y direction. Make a rough plot of $y(x)$.

4.0 solution

We wish to minimize time t, which is

$$t = \int_{x_1}^{x_2} \frac{\mathrm{d}s}{v} = \int_{x_1}^{x_2} \frac{\mathrm{d}s}{\left(\frac{c}{n(x)}\right)} = \frac{1}{c} \int_{x_1}^{x_2} n(x)\,\mathrm{d}s = \frac{k}{c} \int_{x_1}^{x_2} (1 + a\,x) \sqrt{1 + y'^2}\,\mathrm{d}x, \tag{4.1}$$

where k is the proportionality constant for $n(x)$. So we may say

$$f = (1 + a\,x) \sqrt{1 + y'^2}. \tag{4.2}$$

Plugging f into Euler's equation gives

$$0 - \frac{\mathrm{d}}{\mathrm{d}x} \left[(1 + a\,x) \frac{y'}{\sqrt{1 + y'^2}} \right] = 0 \quad \Rightarrow \quad (1 + a\,x) \frac{y'}{\sqrt{1 + y'^2}} = c_1 \quad \Rightarrow \quad (1 + a\,x)\,y' = c_1 \sqrt{1 + y'^2}$$

$$\Rightarrow \quad (1 + a\,x)^2 y'^2 = c_1^2 \left(1 + y'^2\right) \quad \Rightarrow \quad y'^2 \left[(1 + a\,x)^2 - c_1^2\right] = c_1^2 \quad \Rightarrow \quad y'^2 = \frac{c_1^2}{(1 + a\,x)^2 - c_1^2}$$

$$\Rightarrow \quad y' = \frac{c_1}{\sqrt{(1 + a\,x)^2 - c_1^2}} \quad \Rightarrow \quad y = \int \frac{c_1}{\sqrt{(1 + a\,x)^2 - c_1^2}}\,\mathrm{d}x \quad \Rightarrow \quad y = \frac{c_1}{a} \int \frac{\mathrm{d}\,(1 + a\,x)}{\sqrt{(1 + a\,x)^2 - c_1^2}}.$$

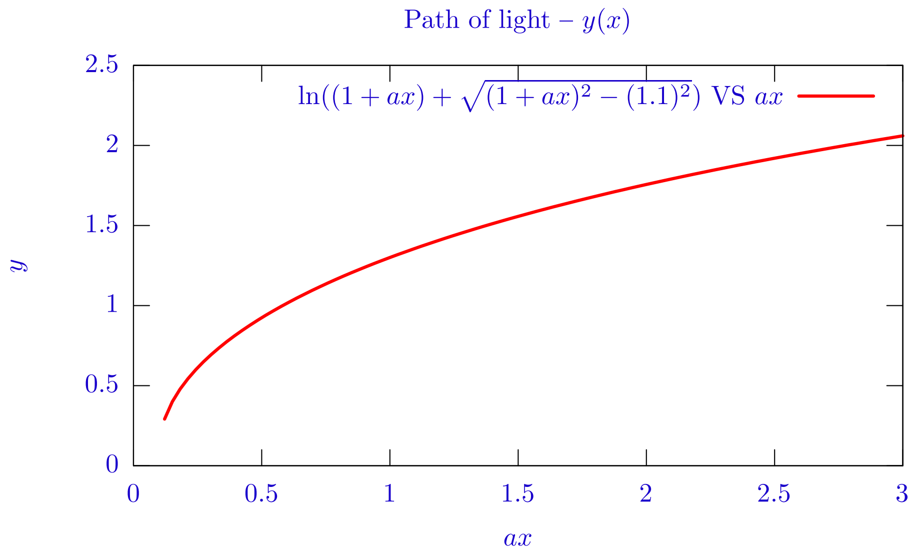
Using the integal

$$\int \frac{\mathrm{d}u}{\sqrt{u^2 - b^2}} = \ln\left(u + \sqrt{u^2 - b^2}\right) \tag{4.3}$$

with $u = 1 + a\,x$ and $b = c_1$, we get

$$\Rightarrow \quad \boxed{y = \frac{c_1}{a} \ln\left[(1 + a\,x) + \sqrt{(1 + a\,x)^2 - c_1^2}\right] + c_2}, \tag{4.4}$$

where we have introduced c_1 and c_2 as the constants of integration.



It looks like $y(x)$ tries to keep the light moving in the smaller x values, where the light travels faster, before moving to the larger x values were the light would travel more slowly. I don't see an obvious way to scale out the c_1 integration constant when plotting this.