1 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \left(\frac{1}{2} \, y'^2 + a \, y \right) \, \mathrm{d}x, \tag{1.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$, and a is a constant.

Hints: Do so by using the Euler equation,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0, \tag{1.2}$$

where $f(y, y'; x) = \frac{1}{2} \, y'^2 + a \, y$. You do not have to determine the two constants of integration, just call them c_1 and c_2.

2 Shortest Line Between Two Points

Show that the shortest line between two points in a plane is a straight line.

3 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2} \, \mathrm{d}x}{x}, \tag{3.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. What common geometry does $y(x)$ represent?

4 Application of Fermat's Principle

Fermat's principle states that light traverses the path between two points which takes the least time.

The index of refraction for light traveling in a medium n is defined as $n \equiv \frac{c}{v}$, where c is the speed of light in a vacuum and v is the speed of light in the medium.

Find the path that light will travel, $y(x)$, in a medium with an index of refraction $n(x)$ that is proportional to $1 + a \, x$, where a is a constant. The light travels in a region of space where x is positive. There is no variation of n along the y direction. Make a rough plot of $y(x)$.