1 Stationary Integral

Find the differential equations that $x(t)$ and $y(t)$ must satisfy such that the following integral is stationary,

$$J = \int_{t_1}^{t_2} \left(\frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\dot{y}^2 - k\,xy + xA\cos\omega t \right) \mathrm{d}t, \tag{1.1}$$

where $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$, $\dot{y} \equiv \frac{\mathrm{d}y}{\mathrm{d}t}$, and k, A, and ω are constants. Hints: Use Euler's equations. You do not need to solve for $x(t)$ and $y(t)$, just find the differential equations that $x(t)$ and $y(t)$ must satisfy.

1.0 solution

For J to be stationary Euler's equations must be satisfied, so

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0, \qquad \text{and} \qquad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0, \tag{1.2}$$

where

$$f(x, \dot{x}, y, \dot{y}; x) = \frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\dot{y}^2 - k\,xy + xA\cos\omega t. \tag{1.3}$$

This gives

$$-ky + A\cos\omega t - \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = 0 \qquad \text{and} \qquad -kx - \frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \boxed{\ddot{x} + ky = A\cos\omega t \qquad \text{and} \qquad \ddot{y} + kx = 0}, \tag{1.4}$$

where $\ddot{x} \equiv \frac{\mathrm{d}^2x}{\mathrm{d}t^2}$ and $\ddot{y} \equiv \frac{\mathrm{d}^2y}{\mathrm{d}t^2}$.