1 Shortest Path in Between Two Points

Show that the shortest path between two points in three-dimensional space is that of a straight line. Hint: Use Euler's equations with several dependent variables $x(t)$, $y(t)$, and $z(t)$, with t being the independent variable.

1.0 solution

We minimize

$$J = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} \tag{1.1}$$

where $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$, are the differentials of the Cartesian coordinates x, y, and z respectively, and x_1 and x_2 correspond to two fixed positions in the Cartesian space. Since $x = x(t)$, $y = y(t)$, and $z = z(t)$, we have

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}t}\,\mathrm{d}t \equiv x'\mathrm{d}t \qquad \mathrm{d}y = \frac{\mathrm{d}y}{\mathrm{d}t}\,\mathrm{d}t \equiv y'\mathrm{d}t \qquad \mathrm{d}z = \frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t \equiv z'\mathrm{d}t. \tag{1.2}$$

So

$$J = \int_{t_1}^{t_2} \sqrt{(x'\,\mathrm{d}t)^2 + (y'\,\mathrm{d}t)^2 + (z'\,\mathrm{d}t)^2} = \int_{t_1}^{t_2} \sqrt{x'^2 + y'^2 + z'^2}\,\mathrm{d}t, \tag{1.3}$$

where t_1 and t_2 are the corresponding values of the variable t at positions x_1 and x_2. We define the functional

$$f = \sqrt{x'^2 + y'^2 + z'^2} \tag{1.4}$$

where we require that it satisfy the following Euler equations

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial x'} = 0 \qquad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial y'} = 0 \qquad \frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial z'} = 0, \tag{1.5}$$

which will make J at least stationary, but we will guess that it's a minimum. Doing this gives

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}\right) = 0 \qquad 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}}\right) = 0 \qquad 0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}}\right) = 0, \tag{1.6}$$

which gives

$$\frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_1 \qquad \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_2 \qquad \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = C_3, \tag{1.7}$$

where C_1, C_2, and C_3, are constants of integration. We may rewrite equations 1.7 as

$$\frac{x'}{C_1} = \sqrt{x'^2 + y'^2 + z'^2} \qquad \frac{y'}{C_2} = \sqrt{x'^2 + y'^2 + z'^2} \qquad \frac{z'}{C_3} = \sqrt{x'^2 + y'^2 + z'^2}. \tag{1.8}$$

So

$$\frac{x'}{C_1} = \frac{y'}{C_2} = \frac{z'}{C_3} \qquad \Rightarrow \qquad \frac{1}{C_1}\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = \frac{1}{C_2}\frac{\mathrm{d}y}{\mathrm{d}t}\mathrm{d}t = \frac{1}{C_3}\frac{\mathrm{d}z}{\mathrm{d}t}\mathrm{d}t \qquad \Rightarrow \qquad \frac{1}{C_1}\mathrm{d}x = \frac{1}{C_2}\mathrm{d}y = \frac{1}{C_3}\mathrm{d}z, \tag{1.9}$$

which may be integrated to give

$$\int \frac{1}{C_1}\mathrm{d}x = \int \frac{1}{C_2}\mathrm{d}y = \int \frac{1}{C_3}\mathrm{d}z \qquad \Rightarrow \qquad \boxed{\frac{1}{C_1}x + C_4 = \frac{1}{C_2}y + C_5 = \frac{1}{C_3}z + C_6}, \tag{1.10}$$

which is a general equation of a straight line, where the integration constants may be adjusted to fit the line to the end points.

2 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \sqrt{x}\sqrt{1+y'^2}\,\mathrm{d}x, \tag{2.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$.

2.0 solution

The functional

$$f = \sqrt{x}\sqrt{1+y'^2} \tag{2.2}$$

must satisfy Euler's equation

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0, \tag{2.3}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. So

$$0 - \frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \quad \Rightarrow \quad \sqrt{x}\frac{y'}{\sqrt{1+y'^2}} = c_1, \tag{2.4}$$

where we introduce the constant of integration c_1. We may rewrite equation 2.4 as

$$\sqrt{x}y' = c_1\sqrt{1+y'^2} \quad \Rightarrow \quad xy'^2 = c_1^2\left(1+y'^2\right) \quad \Rightarrow \quad xy'^2 = c_1^2 + c_1^2 y'^2 \quad \Rightarrow \quad xy'^2 - c_1^2 y'^2 = c_1^2$$

$$\Rightarrow \quad y'^2\left(x-c_1^2\right) = c_1^2 \quad \Rightarrow \quad y'^2 = \frac{c_1^2}{x-c_1^2} \quad \Rightarrow \quad y' = \pm\frac{c_1}{\sqrt{x-c_1^2}} \quad \Rightarrow \quad y = \pm\int\frac{c_1\,\mathrm{d}x}{\sqrt{x-c_1^2}} \tag{2.5}$$

$$\Rightarrow \quad y = \pm c_1\int\frac{\mathrm{d}\left(x-c_1^2\right)}{\sqrt{x-c_1^2}} \quad \Rightarrow \quad \boxed{y = \pm 2c_1\sqrt{x-c_1^2}+c_2}, \tag{2.6}$$

where c_2 is another integration constant.

3 Geodesic: Shortest path on a Circular Cylinder, Three Ways

3.1 Using One Dependent Variable $z(\phi)$

Show that the shortest path between two points on a circular cylinder is along a helix. Do this with the assumption that, $z(\phi)$ may be written as a function of ϕ in cylindrical coordinates (r, ϕ, z) with r being a constant. The equation of a helix in cylindrical coordinates is $z = C_1\,\phi + C_2$ where C_1 and C_2 are constants.

3.1 solution

The length of the line using cylindrical coordinates (r, ϕ, z) can be written as

$$J = \int\sqrt{(\mathrm{d}r)^2 + (r\,\mathrm{d}\phi)^2 + (\mathrm{d}z)^2} = \int\sqrt{(R\mathrm{d}\phi)^2 + (\mathrm{d}z)^2} \tag{3.1}$$

where the integration limits are at two points on the cylinder, and we have imposed the equation of a cylinder $r = R$, where R is the constant radius of the cylinder. Since we only have two variables, z and ϕ, we can choose that the z is the dependent variable, and ϕ is the independent variable giving

$$J = \int \sqrt{(R\,\mathrm{d}\phi)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}\phi}\,\mathrm{d}\phi\right)^2} = \int \sqrt{R^2 + z'^2}\,\mathrm{d}\phi, \tag{3.2}$$

where $z' \equiv \frac{\mathrm{d}z}{\mathrm{d}\phi}$. We make J stationary with Euler's equation,

$$\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}\phi}\frac{\partial f}{\partial z'} = 0 \tag{3.3}$$

where

$$f = \sqrt{R^2 + z'^2}. \tag{3.4}$$

This gives

$$0 - \frac{\mathrm{d}}{\mathrm{d}\phi}\left(\frac{z'}{\sqrt{R^2 + z'^2}}\right) = 0 \quad \Rightarrow \quad \frac{z'}{\sqrt{R^2 + z'^2}} = c_1, \tag{3.5}$$

where c_1 is a constant of integration. Solving for z we get

$$z' = c_1\sqrt{R^2 + z'^2} \quad \Rightarrow \quad z'^2 = c_1^2\left(R^2 + z'^2\right) \quad \Rightarrow \quad z'^2\left(1 - c_1^2\right) = c_1^2 R^2 \quad \Rightarrow \quad z' = \frac{c_1 R}{\sqrt{1 - c_1^2}}$$

$$\Rightarrow \quad \int z'\mathrm{d}\phi = \int \frac{c_1 R}{\sqrt{1 - c_1^2}}\,\mathrm{d}\phi \quad \Rightarrow \quad \int \mathrm{d}z = \frac{c_1 R}{\sqrt{1 - c_1^2}}\,\phi + c_2 \quad \Rightarrow \quad \boxed{z = \frac{c_1 R}{\sqrt{1 - c_1^2}}\,\phi + c_2}, \tag{3.6}$$

which is an equation of a helix.

3.2 With Two Dependent Variables

Show that the shortest path between two points on a circular cylinder is along a helix. Do this using cylindrical coordinates with the variables $(r, \phi(t), z(t))$. That is $\phi(t)$, and $z(t)$ are dependent on an independent variable t, and r being a constant. Note: You do not have to solve the t dependence of ϕ and z.

3.2 solution

The length of the line using cylindrical coordinates (r, ϕ, z) can be written as

$$J = \int \sqrt{(\mathrm{d}r)^2 + (r\,\mathrm{d}\phi)^2 + (\mathrm{d}z)^2} = \int \sqrt{(R\mathrm{d}\phi)^2 + (\mathrm{d}z)^2} \tag{3.7}$$

where the integration limits are at two points on the cylinder, and we have imposed the equation of a cylinder $r = R$, where R is the constant radius of the cylinder. Given that $\phi(t)$, and $z(t)$ are dependent on an independent variable t we have

$$J = \int \sqrt{\left(R\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\mathrm{d}t\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t\right)^2} = \int \sqrt{R^2\phi'^2 + z'^2}\,\mathrm{d}t, \tag{3.8}$$

where $z' \equiv \frac{\mathrm{d}z}{\mathrm{d}t}$, and $\phi' \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t}$. We make J stationary (minimum) with the Euler equations,

$$\frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial z'} = 0 \qquad \frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \phi'} = 0 \tag{3.9}$$

where

$$f = \sqrt{R^2 \phi'^2 + z'^2}. \tag{3.10}$$

This gives

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z'}{\sqrt{R^2\phi'^2 + z'^2}}\right) = 0 \quad \Rightarrow \quad \frac{z'}{\sqrt{R^2\phi'^2 + z'^2}} = c_1 \quad \Rightarrow \quad \frac{z'}{c_1} = \sqrt{R^2\phi'^2 + z'^2}, \tag{3.11}$$

and

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2\phi'}{\sqrt{R^2\phi'^2 + z'^2}}\right) = 0 \quad \Rightarrow \quad \frac{R^2\phi'}{\sqrt{R^2\phi'^2 + z'^2}} = c_2 \quad \Rightarrow \quad \frac{R^2\phi'}{c_2} = \sqrt{R^2\phi'^2 + z'^2}, \tag{3.12}$$

where c_1 and c_2 are constants of integration. Subtracting the last equation in 3.12 from the last equation in 3.11 gives

$$\frac{z'}{c_1} - \frac{R^2\phi'}{c_2} = 0 \quad \Rightarrow \quad \frac{1}{c_1} z'\mathrm{d}t = \frac{1}{c_2} R^2\phi'\mathrm{d}t \quad \Rightarrow \quad \frac{1}{c_1}\int \mathrm{d}z = \frac{1}{c_2} R^2 \int \mathrm{d}\phi \quad \Rightarrow \quad \boxed{z = \frac{c_1}{c_2} R^2\,\phi + c_3}, \tag{3.13}$$

which is an equation of a helix.

3.3 With an Equation of Constraint and Three Dependent Variables

Show that the shortest path between two points on a circular cylinder is along a helix. Do this using cylindrical coordinates with the variables $(r(t), \phi(t), z(t))$ all dependent on an independent variable t, and with the constraint that $r = a$, where a is a constant. Note: There's not much different between this and the last sub-problem.

3.3 solution

The length of the line using cylindrical coordinates (r, ϕ, z) can be written as

$$J = \int \sqrt{(\mathrm{d}r)^2 + (r\,\mathrm{d}\phi)^2 + (\mathrm{d}z)^2} \tag{3.14}$$

where the integration limits are at two points on the cylinder. Given that $r(t)$, $\phi(t)$, and $z(t)$ are dependent on an independent variable t we have

$$J = \int \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}t}\,\mathrm{d}t\right)^2 + \left(r\,\frac{\mathrm{d}\phi}{\mathrm{d}t}\mathrm{d}t\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\,\mathrm{d}t\right)^2} = \int \sqrt{r'^2 + r^2\phi'^2 + z'^2}\,\mathrm{d}t \tag{3.15}$$

where $r' \equiv \frac{\mathrm{d}r}{\mathrm{d}t}$, $\phi' \equiv \frac{\mathrm{d}\phi}{\mathrm{d}t}$, and $z' \equiv \frac{\mathrm{d}z}{\mathrm{d}t}$. We make J stationary (minimum) with the Euler equations,

$$\frac{\partial f}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial r'} + \lambda(t)\frac{\partial g}{\partial r} = 0 \qquad \frac{\partial f}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \phi'} + \lambda(t)\frac{\partial g}{\partial \phi} = 0 \qquad \frac{\partial f}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial z'} + \lambda(t)\frac{\partial g}{\partial z} = 0 \tag{3.16}$$

where $\lambda(t)$ is the Lagrange undetermined multiplier,

$$f = \sqrt{r'^2 + r^2\phi'^2 + z'^2}, \tag{3.17}$$

and we have the equation of constraint $g(r) = r - R = 0$, where R is the constant radius of the cylinder. The equation of constraint $g(r)$ makes solving for $r(t)$ trivial, giving $r(t) = R$ and $r' = 0$. The Euler equations for ϕ and z give

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{R^2\phi'}{\sqrt{R^2\phi'^2 + z'^2}}\right) + 0 = 0 \quad \Rightarrow \quad \frac{R^2\phi'}{\sqrt{R^2\phi'^2 + z'^2}} = c_1 \quad \Rightarrow \quad \frac{R^2\phi'}{c_1} = \sqrt{R^2\phi'^2 + z'^2}, \tag{3.18}$$

and

$$0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{z'}{\sqrt{R^2\phi'^2 + z'^2}}\right) + 0 = 0 \quad \Rightarrow \quad \frac{z'}{\sqrt{R^2\phi'^2 + z'^2}} = c_2 \quad \Rightarrow \quad \frac{z'}{c_2} = \sqrt{R^2\phi'^2 + z'^2}, \tag{3.19}$$

where c_1 and c_2 are constants of integration. Subtracting the last equation in 3.18 from the last equation in 3.19 gives

$$\frac{z'}{c_2} - \frac{R^2\phi'}{c_1} = 0 \quad \Rightarrow \quad \frac{1}{c_2}z'\mathrm{d}t = \frac{1}{c_1}R^2\phi'\mathrm{d}t \quad \Rightarrow \quad \frac{1}{c_2}\int \mathrm{d}z = \frac{1}{c_1}R^2\int \mathrm{d}\phi \quad \Rightarrow \quad \boxed{z = \frac{c_2}{c_1}R^2\,\phi + c_3}, \tag{3.20}$$

which is an equation of a helix.

4 Minimum Area of Revolution

A surface is generated by revolving a curve $y(x)$, that connects two points in the x-y plane, around the x axis. Find this curve such that the surface area is a minimum. Consider using the second form of Euler's equation.

4.0 solution

We minimize

$$J = \int_{x_1}^{x_2} 2\pi y \mathrm{d}s = 2\pi \int_{x_1}^{x_2} y\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = 2\pi \int_{x_1}^{x_2} y\sqrt{1 + y'^2}\mathrm{d}x \tag{4.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$. For J to be a minimum, J must be stationary. J will be stationary if

$$f \equiv y\sqrt{1 + y'^2} \tag{4.2}$$

satisfies Euler's equation,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} = 0. \tag{4.3}$$

Plugging in f into Euler's equation gives

$$\sqrt{1 + y'^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1 + y'^2}}\right) \quad \Rightarrow \quad \sqrt{1 + y'^2} - \frac{yy''}{\sqrt{1 + y'^2}} - \frac{y'^2}{\sqrt{1 + y'^2}} + \frac{yy'^2y''}{(1 + y'^2)^{\frac{3}{2}}} = 0, \tag{4.4}$$

where $y'' \equiv \frac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Multiplying by $\left(1 + y'^2\right)^{\frac{3}{2}}$ gives

$$\left(1 + y'^2\right)^2 - yy''\left(1 + y'^2\right) - y'^2\left(1 + y'^2\right) + yy'^2y'' = 0 \tag{4.5}$$

$$\Rightarrow \quad 1 + 2y'^2 + y'^4 - yy'' - yy'^2y'' - y'^2 - y'^4 + yy'^2y'' = 0 \tag{4.6}$$

$$\Rightarrow \quad 1 + y'^2 - yy'' = 0 \quad \Rightarrow \quad 1 + y'^2 = yy''. \tag{4.7}$$

Assuming that y' can be expressed as a function of y we get,

$$y'' = \frac{\mathrm{d}y'}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y'}{\mathrm{d}y}y'. \tag{4.8}$$

Therefore

$$1+y'^2 = y\frac{\mathrm{d}y'}{\mathrm{d}y}y' \quad \Rightarrow \quad \frac{\mathrm{d}y}{y} = \frac{y'\,\mathrm{d}y'}{1+y'^2} \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{y} = \frac{1}{2}\int \frac{\mathrm{d}\left(1+y'^2\right)}{1+y'^2} \quad \Rightarrow \quad \ln y = \frac{1}{2}\ln\left(1+y'^2\right) + c_1, \tag{4.9}$$

where c_1 is a constant of integration. This gives

$$\ln\left(\frac{y}{\sqrt{1+y'^2}}\right) = c_1 \quad \Rightarrow \quad \frac{y}{\sqrt{1+y'^2}} = C_1, \tag{4.10}$$

where $C_1 \equiv e^{c_1}$. So

$$y = C_1\sqrt{1+y'^2} \quad \Rightarrow \quad y^2 = C_1^2\left(1+y'^2\right) \quad \Rightarrow \quad y^2 = C_1^2 + C_1^2 y'^2 \quad \Rightarrow \quad y^2 - C_1^2 = C_1^2 y'^2 \tag{4.11}$$

$$\Rightarrow \quad \sqrt{y^2 - C_1^2} = C_1 y' \quad \Rightarrow \quad \mathrm{d}x = C_1\frac{y'\,\mathrm{d}x}{\sqrt{y^2-C_1^2}} \tag{4.12}$$

$$\Rightarrow \quad \int \mathrm{d}x = C_1\int\frac{\mathrm{d}y}{\sqrt{y^2-C_1^2}} \quad \Rightarrow \quad x = C_1\cosh^{-1}\frac{y}{C_1} + C_2 \quad \Rightarrow \quad \boxed{y(x) = \cosh\frac{x-C_2}{C_1}}. \tag{4.13}$$

If we used the second form of Euler's equation,

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = 0 \tag{4.14}$$

with f from equation 4.2 we get

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}}\right) = 0 \quad \Rightarrow \quad y\sqrt{1+y'^2} - \frac{yy'^2}{\sqrt{1+y'^2}} = C_1, \tag{4.15}$$

where C_1 is an integration constant. By multiplying by $\sqrt{1+y'^2}$ we get

$$\Rightarrow \quad y + yy'^2 - yy'^2 = C_1\sqrt{1+y'^2} \quad \Rightarrow \quad y = C_1\sqrt{1+y'^2} \tag{4.16}$$

which is the same as the first equation in 4.11. So the result will be the same and we see this way is a little quicker.

5 Shape of a Hanging Cable

A cable with uniform linear mass density, and a fixed length l is hung across space while gravity pulls down on it causing the cable to sag between the two points at the ends of the cable. The total gravitational potential energy (mgy) from the earth pulling down on the cable is a minimum. What is the general shape of the cable, $y(x)$. Hint: Use the method on page 222 of Thorton and Marion.

5.0 solution

Let ρ be the linear mass density of the cable. Letting gravity act in the minus y direction, the gravitational potential energy of the cable is

$$J = \int_{x_1}^{x_2}(\mathrm{d}m)\,gy = \int_{x_1}^{x_2}(\rho\,\mathrm{d}s)\,gy = \rho g\int_{x_1}^{x_2} y\sqrt{1+y'^2}\,\mathrm{d}x. \tag{5.1}$$

The constraint that the cable has a total length l can be written as

$$l = \int_{x_1}^{x_2} \mathrm{d}s = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, \mathrm{d}x. \tag{5.2}$$

We set $f = y\sqrt{1+y'^2}$ and $g = \sqrt{1+y'^2}$ so for J to be a minimum, it must be stationary, and so (from equation 6.78 in Thorton and Marion)

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial f}{\partial y'} + \lambda\left(\frac{\partial g}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial g}{\partial y'}\right) = 0 \tag{5.3}$$

where λ is a Lagrange undetermined multiplier constant. This gives

$$\sqrt{1+y'^2} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{yy'}{\sqrt{1+y'^2}}\right) - \lambda\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0.$$

$$\Rightarrow \quad \sqrt{1+y'^2} - \frac{y'^2}{\sqrt{1+y'^2}} - y\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) - \lambda\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+y'^2}} - (y+\lambda)\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y'}{\sqrt{1+y'^2}}\right) = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+y'^2}} - (y+\lambda)\left(\frac{y''}{\sqrt{1+y'^2}} - \frac{y'^2 y''}{\left(1+y'^2\right)^{\frac{3}{2}}}\right) = 0.$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+y'^2}} - (y+\lambda)\frac{y''\left(1+y'^2\right) - y'^2 y''}{\left(1+y'^2\right)^{\frac{3}{2}}} = 0 \quad \Rightarrow \quad \frac{1}{\sqrt{1+y'^2}} - (y+\lambda)\frac{y'' + y'^2 y'' - y'^2 y''}{\left(1+y'^2\right)^{\frac{3}{2}}} = 0$$

$$\Rightarrow \quad \frac{1}{\sqrt{1+y'^2}} - (y+\lambda)\frac{y''}{\left(1+y'^2\right)^{\frac{3}{2}}} = 0 \tag{5.4}$$

Multiplying by $\left(1+y'^2\right)^{\frac{3}{2}}$ gives

$$\Rightarrow \quad 1 + y'^2 - (y+\lambda)\,y'' = 0 \quad \Rightarrow \quad 1 + y'^2 = (y+\lambda)\,y'' \quad \Rightarrow \quad 1 + y'^2 = (y+\lambda)\frac{\mathrm{d}y'}{\mathrm{d}y}y'$$

$$\Rightarrow \quad \frac{\mathrm{d}y}{y+\lambda} = \frac{y'\,\mathrm{d}y'}{1+y'^2} \quad \Rightarrow \quad \int\frac{\mathrm{d}y}{y+\lambda} = \int\frac{y'\,\mathrm{d}y'}{1+y'^2} \quad \Rightarrow \quad \int\frac{\mathrm{d}\,(y+\lambda)}{y+\lambda} = \frac{1}{2}\int\frac{\mathrm{d}\left(1+y'^2\right)}{1+y'^2}. \tag{5.5}$$

We note that we just solved an equation like this from the previous problem in 4.9, but with $y + \lambda \to y$, and so the solution, from equation 4.13, is

$$y + \lambda = \cosh\frac{x - C_2}{C_1} \quad \Rightarrow \quad \boxed{y(x) = -\lambda + \cosh\frac{x - C_2}{C_1}}, \tag{5.6}$$

so the general shape is a hyperbolic cosine function.