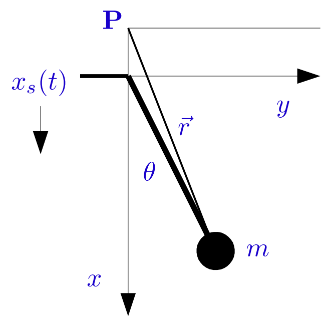
1 Pendulum with Vertically Moving Pivot

A frictionless pendulum swings in the x-y plane. The x direction is down in the direction of the uniform gravitational field g. The y direction is to the right. The pendulum bob has a mass of m. The pendulum has a length l. The position of the pivot point of the pendulum is moving (up and down) along the x-direction with the pivot position given by $x_s(t)$.

1.1 Lagrangian

Find the Lagrangian for this system $L(\theta, \dot\theta, t)$ where θ is the angle that the pendulum makes with the (vertical) x axis toward the y axis. Answer in terms of m, g, l, $x_s(t)$, $\dot x_s(t)$, θ, and $\dot\theta$. You may consider $x_s(t)$ and $\dot x_s(t)$ as given functions of t, so $L(\theta, \dot\theta, t) = L(\theta, \dot\theta, x_s(t), \dot x_s(t))$

1.1 solution



In order to find T, the kinetic energy, we'll first calculate v^2, the speed squared for the bob, as a function of θ and $\dot\theta$. Using the figure above we get the position of the bob, $\vec r$, relative to the point **P** which is not moving.

$$\vec r = (x + x_s)\,\hat x + y\,\hat y = (l\cos\theta + x_s)\,\hat x + l\sin\theta\,\hat y \quad \Rightarrow \quad \dot{\vec r} = \left(-l\dot\theta\sin\theta + \dot x_s\right)\,\hat x + l\dot\theta\cos\theta\,\hat y \tag{1.1}$$

$$\Rightarrow \quad v^2 = \left(\dot{\vec r}\right)^2 = \left(-l\dot\theta\sin\theta + \dot x_s\right)^2 + \left(l\dot\theta\cos\theta\right)^2 = -2l\dot x_s\sin\theta + \dot x_s^2 + l^2\dot\theta^2\sin^2\theta + l^2\dot\theta^2\cos^2\theta \tag{1.2}$$

$$= -2l\dot x_s\dot\theta\sin\theta + \dot x_s^2 + l^2\dot\theta^2\left(\sin^2\theta + \cos^2\theta\right) = -2l\dot x_s\dot\theta\sin\theta + \dot x_s^2 + l^2\dot\theta^2. \tag{1.3}$$

So

$$L = T - U = \frac{1}{2}mv^2 - \left(-mg\,(x + x_s)\right) = \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}m\dot x_s^2 - ml\dot x_s\dot\theta\sin\theta + mgl\cos\theta + mgx_s \tag{1.4}$$

$$\Rightarrow \quad \boxed{L = \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}m\dot x_s^2(t) - ml\dot x_s(t)\,\dot\theta\sin\theta + mgl\cos\theta + mgx_s(t)} \tag{1.5}$$

1.2 Equations of Motion

Using this Lagrangian find the equations of motion for θ (something like $\ddot\theta = ?$). Answer in terms of m, g, l, $x_s(t)$, $\dot x_s(t)$, $\ddot x_s(t)$, θ, $\dot\theta$, and $\ddot\theta$.

1.2 solution

$$\frac{\partial L}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial L}{\partial\dot\theta} = 0 \quad \Rightarrow \quad -ml\dot x_s\dot\theta\cos\theta - mgl\sin\theta - \frac{\mathrm{d}}{\mathrm{dt}}\left(ml^2\dot\theta - ml\dot x_s\sin\theta\right)$$

$$\Rightarrow \quad -ml\dot{x}_s\dot{\theta}\cos\theta - mgl\sin\theta - ml^2\ddot{\theta} + ml\ddot{x}_s\sin\theta + ml\dot{x}_s\dot{\theta}\cos\theta = 0$$

$$\Rightarrow \quad ml^2\ddot{\theta} = -mgl\sin\theta + ml\ddot{x}_s\sin\theta$$

$$\Rightarrow \quad \boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{\ddot{x}_s(t)}{l}\sin\theta}. \tag{1.6}$$