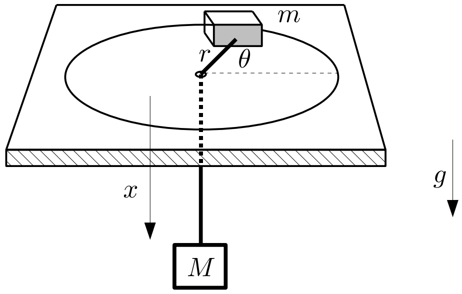
1 Spinning Block and Hanging Weight using Lagrange Multiplier



A block, with mass m, slides without friction on a level table. A stretch-less, massless string connects this block to a hanging weight, with mass M, through a hole in the table. The block on the table can orbit around the table so that the angle θ and the radial distance r both change with time. When r changes the distance from the table to the hanging weight, x, changes too. This gives us the constraint equation $f = r + x - a = 0$, where a is a constant. Consider your dynamical (generalized coordinate) variables to be θ, r, and x.

1.1 Find Equations of Motion

Find the equations of motion for θ, r, and x using a Lagrange multiplier λ, with the given constraint equation. You do not have to solve them, or solve for λ.

1.1 solution

The Lagrangian is

$$L = T - U = T_M + T_m - U_M - U_m = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - Mg\,(-x) - 0 \tag{1.1}$$

$$\Rightarrow \quad L\left(\theta, r, x, \dot{\theta}, \dot{r}, \dot{x}\right) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + Mgx. \tag{1.2}$$

Lagrange's equations with an undetermined multiplier λ for this system gives

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0 \quad \frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} + \lambda\frac{\partial f}{\partial r} = 0 \qquad \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial f}{\partial x} = 0. \tag{1.3}$$

This gives us

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0} \tag{1.4}$$

$$mr\dot{\theta}^2 - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r}\right) + \lambda\,[1] = 0 \quad \Rightarrow \quad \boxed{m\ddot{r} = mr\dot{\theta}^2 + \lambda} \tag{1.5}$$

$$Mg - \frac{\mathrm{d}}{\mathrm{d}t}\left(M\dot{x}\right) + \lambda\,[1] = 0 \quad \Rightarrow \quad \boxed{M\ddot{x} = Mg + \lambda} \tag{1.6}$$

and the equation of constraint gives

$$\boxed{\ddot{r} + \ddot{x} = 0}. \tag{1.7}$$

1.2 Remove the $\dot\theta$ Dependence in the r Equation

The integration constant for the θ equation is the angular momentum of the block about the hole, l. Find the first integral of the θ equation and remove the $\dot\theta$ in the r equation of motion.

1.2 solution

From equation 1.4

$$\boxed{mr^2\dot\theta = l}\,. \tag{1.8}$$

Plugging $\dot\theta$ into equation 1.5 gives

$$m\ddot r = mr\left(\frac{l}{mr^2}\right)^2 + \lambda \quad \Rightarrow \quad \boxed{m\ddot r = \frac{l^2}{mr^3} + \lambda}\,. \tag{1.9}$$

1.3 Lagrange Multiplier, λ

Physically what is the Lagrange multiplier, λ? In general, is it constant or a function of time? Justify your answer.

1.3 solution

Computing the force associated with the constraint of the string we get

$$\mathcal{Q}_x = \lambda\frac{\partial f}{\partial x} = \lambda\,[1] = \lambda\,, \tag{1.10}$$

as we computed before in equation 1.6. λ is a force of constraint from the string. This force is in the plus x direction which is opposite the direction of the tension in the string, so $\boxed{\lambda \text{ is minus the tension in the string}}$.

We can put together all the equations of motion (equation 1.5 minus equation 1.6 and setting $\ddot x$ to $-\ddot r$ from equation 1.7) to get

$$(M+m)\,\ddot r = \frac{l^2}{mr^3} - Mg \tag{1.11}$$

which shows that $\ddot r$ does not have to be constant and if $\ddot r$ is not constant neither is $\ddot x$, and from equation 1.6 neither is λ. So $\boxed{\lambda \text{ is a function of time in general}}$.