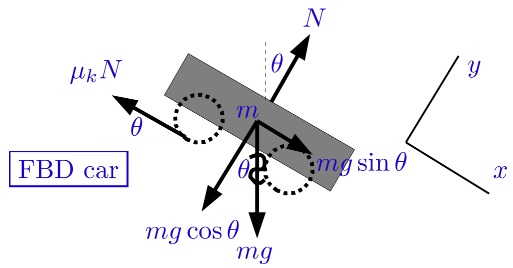
1 Sliding Car

A car traveling down an incline with a 8% grade (raise/run) locks his brakes and skids 30 m before hitting a parked car. The coefficient of kinetic friction between the tires and the road is $\mu_k = 0.45$. Was the car exceeding the 25 MPH speed limit? Explain.

1.0 solution



The above figure shows a free body diagram (FBD) of the car. We let m be the mass of the car. Applying Newton's 2nd law gives

$$\sum F_y = N - mg\cos\theta = 0 \quad \Rightarrow \quad N = mg\cos\theta \tag{1.1}$$

$$\sum F_x = mg\sin\theta - \mu_k N = ma_x \quad \Rightarrow \quad ma_x = mg\sin\theta - \mu_k mg\cos\theta = mg\left(\sin\theta - \mu_k\cos\theta\right)$$

$$\Rightarrow \quad a_x = g\left(\sin\theta - \mu_k\cos\theta\right) \tag{1.2}$$

The car has constant acceleration in the x direction. So

$$a_x = \frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{\mathrm{d}v_x}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}v_x}{\mathrm{d}x}v_x \tag{1.3}$$

$$\Rightarrow \quad v_x \mathrm{d}v_x = a_x \mathrm{d}x \quad \Rightarrow \quad \frac{1}{2}v_x^2 = a_x x + c_1\,, \tag{1.4}$$

where c_1 is a constant. Given we set $x = 0$ when $v_x = v_{x,i}$, the initial speed, we find that the speed of the car along the x direction, v_x, as a function of the change in position x can be written as

$$v_x^2 = v_{x,i}^2 + 2a_x\, x. \tag{1.5}$$

By setting $v_{x,i}$ =30 mi/hr, $v_x = 0$, $\theta = \tan^1 0.08$, then x should be the distance the car would travel if the car was going at the speed limit. With $\theta = \tan^1 0.08$, $\sin\theta = \frac{0.08}{\sqrt{1+(0.08)^2}} \approx 0.0797452$, and $\cos\theta = \frac{1}{\sqrt{1+(0.08)^2}} \approx 0.996815$

$$x = \frac{v_{x,i}^2}{2a_x} = -\frac{\left(30\,\frac{\mathrm{mi}}{\mathrm{hour}}\,\frac{5280\mathrm{ft}}{\mathrm{mi}}\,\frac{12\mathrm{in}}{\mathrm{ft}}\,\frac{0.0254\mathrm{m}}{\mathrm{in}}\,\frac{1\mathrm{hour}}{3600\mathrm{s}}\right)^2}{2\,\left(9.8\frac{\mathrm{m}}{\mathrm{s}^2}\right)\left[0.0797452 - (0.45)\,0.996815\right]} \approx 17.28\,\mathrm{m}. \tag{1.6}$$

Therefore, $\boxed{\text{he was speeding.}}$ So if he was going at 25 MPH he would have only skidded 17.28 meters. If he skidded 30 meters and than hit a parked car he had to have been traveling initially at at a higher speed. From equation 1.5 with $v_x = 0$ and $a_x < 0$ we see that $v_{x,i} \propto \sqrt{x}$, so $v'_{x,i} = v_{x,i}\sqrt{\frac{x'}{x}} = 25\,\mathrm{MPH}\sqrt{\frac{30}{17.27}} = 32.9\,\mathrm{MPH}$. So he was traveling at 32.9 MPH or faster.

2 Grandfather Clock

A grandfather clock has a pendulum length of 0.7 m and a bob mass of 0.4 kg. A weight of mass 2 kg falls 0.8 m in seven days to keep the amplitude (from equilibrium) of the pendulum oscillating steady at 0.03 rad. What is the quality factor, $\mathcal{Q}$, of this clock? Assume that all the energy is lost in the oscillating pendulum.

2.0 solution

The amplitude of the swing is small so we approximate the pendulum as having simple harmonic motion with angular frequency of $\omega_0 = \frac{g}{l}$. The definition of $\mathcal{Q}$ is

$$\mathcal{Q} = \frac{\omega_R}{2\beta}, \tag{2.1}$$

and

$$\omega_R = \sqrt{\omega_0^2 - \beta^2} \approx \omega_0 = \sqrt{\frac{g}{l}}, \tag{2.2}$$

where $g \approx 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}$ is the acceleration so to gravity at the surface of the earth, l is the length of the pendulum, and we have assumed that β is small compared to ω_0. We'll check that later. So assuming β is small compared to ω_0 we wish to calculate

$$\mathcal{Q} = \frac{1}{2\beta}\sqrt{\frac{g}{l}}, \tag{2.3}$$

We can get β from the rate at which energy is being added to the pendulum by the falling weight. We start with the energy in the pendulum E being

$$E = T + U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1-\cos\theta\right) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\left(1 - 1 + \frac{1}{2}\theta^2\right) = \frac{1}{2}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\theta^2, \tag{2.4}$$

where we have used the small angle approximation for θ. The rate at which the energy in the pendulum changes, which is the power to the pendulum, is

$$\frac{\mathrm{d}E}{\mathrm{d}t} = ml^2\dot{\theta}\ddot{\theta} + mgl\theta\dot{\theta} = ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\theta\right). \tag{2.5}$$

The equation of motion of the pendulum, that defines β in our case, is

$$\ddot{\theta} + \frac{g}{l}\theta + 2\beta\dot{\theta} = A\cos\omega t \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{l}\theta = -2\beta\dot{\theta} + A\cos\omega t \tag{2.6}$$

where A is the amplitude of the driving force, ω is the driving angular frequency, and t is time. Given this we may rewrite equation 2.5 as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = ml^2\dot{\theta}\left(-2\beta\dot{\theta} + A\cos\omega t\right) = -2ml^2\beta\dot{\theta}^2 + ml^2\dot{\theta}A\cos\omega t\,. \tag{2.7}$$

The first term is the rate at which energy is being taken away due to friction (β), and the second term is the power from the driver, P_{in}. The average power, over a period, into the pendulum does not change so we have

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{ave}} = 0 = -2ml^2\beta\,\dot{\theta}^2_{\mathrm{ave}} + P_{\mathrm{in,ave}} \quad \Rightarrow \quad 2ml^2\beta\dot{\theta}^2_{\mathrm{ave}} = P_{\mathrm{in,ave}}\,, \tag{2.8}$$

where $P_{\mathrm{in,ave}}$ is the average power from the driver. At steady state

$$\theta(t) = D\cos\left(\omega t - \delta\right), \tag{2.9}$$

where D is the amplitude of the pendulum motion, ω is the driving angular frequency, and δ is the phase constant. Therefore

$$\dot{\theta}(t) = \omega D\cos\left(\omega t - \delta\right) \quad \Rightarrow \quad \dot{\theta}^2(t) = \omega^2 D^2\cos^2\left(\omega t - \delta\right) \quad \Rightarrow \quad \dot{\theta}^2_{\mathrm{ave}} = \frac{1}{2}\omega^2 D^2 \tag{2.10}$$

So with equation 2.8 we have

$$2ml^2\beta\left(\frac{1}{2}\omega^2 D^2\right) = P_{\mathrm{in,ave}} = \frac{Mgh}{T} \quad \Rightarrow \quad \beta = \frac{1}{ml^2\omega^2 D^2}\frac{Mgh}{T} \approx \frac{1}{ml^2\omega_0^2 D^2}\frac{Mgh}{T} = \frac{Mh}{mlD^2T} \tag{2.11}$$

where we have used the given information about the driving weight in the clock, M is the mass of the falling weight, h is the height that the weight falls in time T, and we assumed that $\omega^2 \approx \omega_0^2$. We see that $\beta^2 \approx 0.0002/s^2$ is much smaller than $\omega_0^2 \approx 14/s^2$, and so our begining assumption in equation 2.2 was consistent. Putting it all together we get

$$\mathcal{Q} \approx \frac{1}{2\beta}\sqrt{\frac{g}{l}} = \frac{1}{2}\left(\frac{Mh}{mlD^2T}\right)^{-1}\sqrt{\frac{g}{l}} = \frac{mTD^2}{2Mh}\sqrt{gl} = \frac{(0.4\text{kg})\,(7\;3600\;24\,s)\,(0.03)^2\,\sqrt{\left(9.8\,\frac{\text{m}}{\text{s}^2}\right)(0.7\,\text{m})}}{2\,(2\,\text{kg})\,(0.8\,\text{m})} = 178.2\,. \quad (2.12)$$

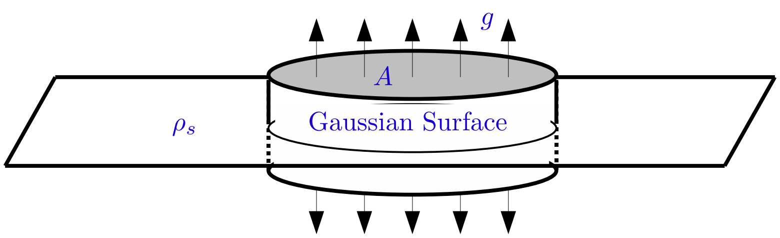
So

$$\boxed{\mathcal{Q} \approx 178}\,. \quad (2.13)$$

3 Gravitation

A uniform solid sphere of mass M and a radius R is fixed a distance h above a thin infinite sheet ofq mass density ρ_s (mass/area). h is greater than R. What is the force on the sheet from the sphere?

3.0 solution



Applying the integral form of Gauss's law to the infinite sheet of mass density ρ_s we get

$$\oint \vec{g}\cdot \mathrm{d}\vec{A} = -4\pi G\rho_s A = (g)\,(2A) \quad \Rightarrow \quad g = -2\pi G\rho_s\,, \quad (3.1)$$

where g is the magnitude of the gravitational field from the infinite sheet, and A is the surface area of the Gaussian that is above and below the sheet. The force on the sheet from the sphere will be equal and opposite to the force on the sphere from the sheet. The magnitude of that force is

$$F_g = Mg = \boxed{2\pi\rho_s GM}\,. \quad (3.2)$$

You'll get the same result by integrating the force from a small piece of the sheet $\mathrm{d}\vec{F}$ over the infinite sheet. It's more work. You may consider that the sphere acts make a point particle mass.

4 A Particle in a Cone

A particle, with mass m, is constrained to move on the surface of a cone. The cone has it's vertex pointing down in the direction of gravity (g). The cone has a half-angle α.

4.1 Lagrangian

Write the Lagrangian, $L\left(r, \phi, \dot{r}, \dot{\phi}\right)$, in terms of spherical polar coordinates r, and ϕ, where the θ coordinate is fixed at value α on the surface of the cone.

4.1 solution

The motion is constrained so that $\theta = \alpha$ so $\dot{\theta} = 0$.

$$L = T - U = \frac{1}{2}m\left[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r(\sin\theta)\dot{\phi}\hat{\phi}\right]\cdot\left[\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r(\sin\theta)\dot{\phi}\hat{\phi}\right] - mg\,(z) = \frac{1}{2}m\left[\dot{r}^2 + r^2(\sin^2\alpha)\dot{\phi}^2\right] - mg\,(r\cos\alpha)$$

$$\Rightarrow \quad \boxed{L\left(r, \phi, \dot{r}, \dot{\phi}\right) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2(\sin^2\alpha)\dot{\phi}^2 - mgr\cos\alpha}. \tag{4.1}$$

4.2 Equations of Motion

Find the equations of motion for r and ϕ. Interpret the ϕ equation in terms of the angular momentum along the z direction, l_z. Use l_z to eliminate the $\dot{\phi}$ from the r equation of motion.

4.2 solution

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = 0 \quad \Rightarrow \quad mr(\sin^2\alpha)\dot{\phi}^2 - mg\cos\alpha - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = 0 \quad \Rightarrow \quad \boxed{\ddot{r} = r(\sin^2\alpha)\dot{\phi}^2 - g\cos\alpha} \tag{4.2}$$

$$\frac{\partial L}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\phi}} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2(\sin^2\alpha)\dot{\phi}\right) = 0 \quad \Rightarrow \quad \boxed{mr^2\left(\sin^2\alpha\right)\dot{\phi} = l_z}. \tag{4.3}$$

This gives

$$\ddot{r} = r\left(\sin^2\alpha\right)\left(\frac{l_z}{mr^2\sin^2\alpha}\right)^2 - g\cos\alpha \quad \Rightarrow \quad \boxed{\ddot{r} = \frac{l_z^2}{m^2r^3\sin^2\alpha} - g\cos\alpha} \tag{4.4}$$

4.3 Find an Equilibrium r Position

Find the equilibrium r position, r_0. Determine if this equilibrium r position is stable or not. If this position is stable, find the frequency of oscillation about this equilibrium position.

4.3 solution

At equilibrium $r \ddot{=} r_0 = 0$ which gives

$$0 = \frac{l_z^2}{m^2r_0^3\sin^2\alpha} - g\cos\alpha \quad \Rightarrow \quad \boxed{r_0 = \sqrt[3]{\frac{l_z^2}{m^2g\sin^2\alpha\,\cos\alpha}}} \tag{4.5}$$

If this equilibrium r position is stable then

$$\left.\frac{\mathrm{d}\ddot{r}}{\mathrm{d}r}\right|_{r=r_0} < 0. \tag{4.6}$$

We have

$$\frac{\mathrm{d}\ddot{r}}{\mathrm{d}r} = -3\frac{l_z^2}{m^2 r^4 \sin^2\alpha} \quad \Rightarrow \quad \left.\frac{\mathrm{d}\ddot{r}}{\mathrm{d}r}\right|_{r=r_0} = -3\frac{l_z^2}{m^2\left(\frac{l_z^2}{m^2 g \sin^2\alpha\,\cos\alpha}\right)^{\frac{4}{3}}\sin^2\alpha} = -3\sqrt[3]{\frac{m^2 g^4 \sin^2\alpha\,\cos^4\alpha}{l_z^2}} \tag{4.7}$$

So it is stable and the angular frequency of oscillation, ω_0, is given by

$$-\omega_0^2 = \left.\frac{\mathrm{d}\ddot{r}}{\mathrm{d}r}\right|_{r=r_0} \quad \Rightarrow \quad \boxed{\omega_0 = \sqrt{3}\,\sqrt[3]{\frac{mg^2\sin\alpha\,\cos^2\alpha}{l_z}}}. \tag{4.8}$$

5 Non-unique Lagrangian

Show that the if a Lagrangian $L(q_1, ..., q_s, \dot{q}_1 ..., \dot{q}_s, t)$ is related to another Lagrangian $L'(q_1, ..., q_s, \dot{q}_1 ..., \dot{q}_s, t)$ by $L' = L + \frac{\mathrm{d}F}{\mathrm{d}t}$, where $F = F(q_1, ..., q_s, t)$, then the two Lagrangians will give exactly the same equations of motion.

5.0 solution

The equation of motion for q_i from L' is

$$\begin{aligned}
&\frac{\partial L'}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L'}{\partial \dot{q}_i} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial q_i}\left(L + \frac{\mathrm{d}F}{\mathrm{d}t}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_i}\left(L + \frac{\mathrm{d}F}{\mathrm{d}t}\right) = 0 \\
&\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + \frac{\partial}{\partial q_i}\frac{\mathrm{d}F}{\mathrm{d}t} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_i}\frac{\mathrm{d}F}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial q_i}F - \frac{\partial}{\partial \dot{q}_i}\frac{\mathrm{d}F}{\mathrm{d}t}\right) = 0 \\
&\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial q_i}F - \frac{\partial}{\partial \dot{q}_i}\left(\sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial t}\right)\right] = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} + \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial F}{\partial q_i} - \frac{\partial F}{\partial q_i}\right] = 0 \\
&\Rightarrow \quad \frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0\,,
\end{aligned} \tag{5.1}$$

which is the same as the equation of motion for L.