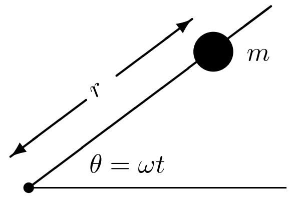
1 Bead on a Spinning Wire

This walks you through the problem.



A bead, with mass m, slides without friction on a long straight wire. The wire is spinning about one end at a constant angular speed ω. Let r be the position of the bead on the wire as measured from the point of rotation of the wire. There is no gravity. All the motion is in a plane.

1.1 Lagrangian

In terms of r, $\dot{r}$, m, ω, and t, write an expression for the Lagrangian $L(r, \dot{r}, t)$ for this system. Note that $U = 0$.

1.1 solution

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right) - 0 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta} \quad \Rightarrow \quad \boxed{L(r, \dot{r}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2}. \tag{1.1}$$

1.2 Lagrange's Equation of Motion

Using your Lagrangian, find the equation of motion for r.

1.2 solution

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = 0 \quad \Rightarrow \quad mr\omega^2 - \frac{\mathrm{d}}{\mathrm{d}t}m\dot{r} = 0 \quad \Rightarrow \quad \boxed{\ddot{r} = r\omega^2}. \tag{1.2}$$

1.3 Canonical Conjugate Momentum

In general, the canonical momentum that is conjugate (paired) to generalized position q_i is $p_{q_i} \equiv p_i = \frac{\partial L}{\partial \dot{q}_i}$.

The canonical momentum that is conjugate to r is $p_r \equiv \frac{\partial L}{\partial \dot{r}}$. Find the canonical momentum that is conjugate to r, p_r, as a function of r, $\dot{r}$, ω, and t.

1.3 solution

$$p_r \equiv \frac{\partial L}{\partial \dot{r}} = \frac{\partial}{\partial \dot{r}}\left(\frac{1}{2}m\dot{r}^2\right) \quad \Rightarrow \quad \boxed{p_r = m\dot{r}}. \tag{1.3}$$

1.4 Hamiltonian

The Hamiltonian in general can be defined as

$$H(q_k, p_k, t) = \sum_j p_j \dot{q}_j - L(q_k, \dot{q}_k, t) \ , \tag{1.4}$$

where the Hamiltonian must be written as a function of just the dependent dynamical variables q_k, p_k, and t. There must be no $\dot{q}_k$ dependence in H. So we are assuming that $\dot{q}_k$ can be written as a function of q_k, p_k, and t, in order to make this transformation.

In our case $H(r, p_r) = p_r \dot{r} - L(r, \dot{r})$. In terms of r, p_r, m, and ω, find the Hamiltonian, $H(r, p_r, t)$ for this system.

1.4 solution

From this definition of H we get

$$H = p_r \dot{r} - \left(\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \omega^2\right) = p_r \left(\frac{p_r}{m}\right) - \frac{1}{2} m \left(\frac{p_r}{m}\right)^2 - \frac{1}{2} m r^2 \omega^2 \quad \Rightarrow \quad \boxed{H(r, p_r) = \frac{p_r^2}{2m} - \frac{1}{2} m r^2 \omega^2}. \tag{1.5}$$

1.5 Hamilton's Equations of Motion

In Lagrangian dynamics, for each generalized position, q_i, $(i = 1, 2, ...s)$ there is an equation of motion that is a second-order ordinary differential equation. In Hamiltonian Dynamics, for each corresponding second-order differential equation of Lagrangian dynamics there are two first-order ordinary differential equations, one of the generalized position, q_i, and one for the corresponding canonical momentum, $p_{q_i} \equiv p_i$. These Hamiltonian equations of motion are gotten from the Hamiltonian like so

$$\dot{q}_i = \frac{\partial H}{\partial p_{q_i}} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{1.6}$$

Use the Hamiltonian that you just calculated and equations 1.6 to find the equations of motion for r and p_r, and show that these equations of motion for r and p_r are equivalent to the equation of motion from section 1.2.

1.5 solution

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \quad \Rightarrow \quad \boxed{\dot{r} = \frac{p_r}{m}}, \tag{1.7}$$

and

$$\dot{p}_r = -\frac{\partial H}{\partial r} = -\left(-m r \omega^2\right) \quad \Rightarrow \quad \boxed{\dot{p}_r = m r \omega^2}. \tag{1.8}$$

From equation 1.7 and equation 1.8

$$\ddot{r} = \frac{1}{m} \dot{p}_r = \frac{1}{m}\left(m r \omega^2\right) = r \omega^2 \quad \Rightarrow \quad \ddot{r} = r \omega^2 \ , \tag{1.9}$$

which is equation 1.2. So the equations of motion from Lagrangian dynamics and Hamiltonian Dynamics are equivalent.

1.6 Energy vs Hamiltonian

In terms of r, p_r, m, ω, and t, **(a)** find the difference between the total energy E and the Hamiltonian H for this system $(E - H)$. **(b)** Is H constant in time? **(c)** Is E constant in time? Note that $E = T = L$. Explain you results.

1.6 solution

$$E - H = \left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2\right) - \left(\frac{p_r^2}{2m} - \frac{1}{2}mr^2\omega^2\right) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 - \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 = mr^2\omega^2$$

$$\Rightarrow \quad \boxed{(\mathbf{a})\; E - H = mr^2\omega^2}. \tag{1.10}$$

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p_r^2}{2m} - \frac{1}{2}mr^2\omega^2\right) = \frac{p_r}{m}\dot{p}_r - m\omega^2 r\dot{r} = \frac{p_r}{m}\left(mr\omega^2\right) - m\omega^2 r\left(\frac{p_r}{m}\right) = 0\,, \tag{1.11}$$

where we have used the equations of motion for $\dot{p}_r = mr\omega^2$ and $\dot{r} = \frac{p_r}{m}$. So $\boxed{\textbf{(b)}\; H \text{ is constant}}$.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2\right) = \frac{p_r}{m}\dot{p}_r + m\omega^2 r\dot{r} = 2rp_r\omega^2\,, \tag{1.12}$$

So $\boxed{\textbf{(c)}\; E \text{ is not constant}}$.