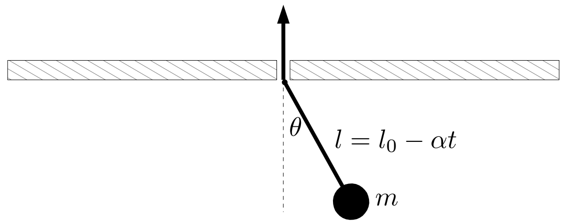
1 Shrinking Pendulum



A simple plane pendulum consists of a mass m attached to a massless string of length l. The suspension point of the pendulum remains fixed as the length, l, is shortened at a constant rate α, as shown above. The pendulum has a length l_0 at time $t = 0$. In this problem we'll find the equations of motion for the pendulum using the Lagrangian and the Hamiltonian method.

1.1 Lagrangian

In terms of θ $\dot{\theta}$, m, α, l_0, and t, write an expression for the Lagrangian, $L\left(\theta, \dot{\theta}, t\right)$, for this system.

1.1 solution

$$L = T - U = \frac{1}{2}m\dot{l}^2 + \frac{1}{2}ml^2\dot{\theta}^2 - mg\left(-l\cos\theta\right) \quad \Rightarrow \quad \boxed{L\left(\theta, \dot{\theta}, t\right) = \frac{1}{2}m\alpha^2 + \frac{1}{2}m\left(l_0 - \alpha t\right)^2\dot{\theta}^2 + mg\left(l_0 - \alpha t\right)\cos\theta} \tag{1.1}$$

1.2 Lagrange's Equations of Motion

Using your Lagrangian and Largrange's equatons, find the differential equation of motion for θ.

1.2 solution

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}L}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad -mg\left(l_0 - \alpha t\right)\sin\theta - \frac{\mathrm{d}L}{\mathrm{d}t}\left[m\left(l_0 - \alpha t\right)^2\dot{\theta}\right] = 0$$

$$\Rightarrow \quad -mg\left(l_0 - \alpha t\right)\sin\theta - 2m\left(l_0 - \alpha t\right)\left(-\alpha\right)\dot{\theta} - m\left(l_0 - \alpha t\right)^2\ddot{\theta} = 0\,. \tag{1.2}$$

Dividing by $m\left(l_0 - \alpha t\right)$ and rearranging gives

$$\Rightarrow \quad \boxed{\left(l_0 - \alpha t\right)\ddot{\theta} = 2\alpha\dot{\theta} - g\sin\theta} \tag{1.3}$$

1.3 Canonical Conjugate Momentum

Find the canonical momentum that is conjugate (paired) to θ. That is, find $p_\theta \equiv \frac{\partial L}{\partial \dot{\theta}}$, as a function of θ $\dot{\theta}$, m, α, l_0, and t.

1.3 solution

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m\left(l_0 - \alpha t\right)^2 \dot{\theta} \quad \Rightarrow \quad \boxed{p_\theta = m\left(l_0 - \alpha t\right)^2 \dot{\theta}} \tag{1.4}$$

1.4 Hamiltonian

In terms of θ, p_θ, m, α, l_0, and t, write the Hamiltonian, $H(\theta, p_\theta, t)$ for this system.

1.4 solution

$$H = p_\theta \dot{\theta} - L = \left[m\left(l_0 - \alpha t\right)^2 \dot{\theta}\right] \dot{\theta} - \left[\frac{1}{2} m\alpha^2 + \frac{1}{2} m\left(l_0 - \alpha t\right)^2 \dot{\theta}^2 + mg\left(l_0 - \alpha t\right) \cos\theta\right]$$

$$= \frac{1}{2} m\left(l_0 - \alpha t\right)^2 \dot{\theta}^2 - \frac{1}{2} m\alpha^2 - mg\left(l_0 - \alpha t\right) \cos\theta \quad \Rightarrow \quad \boxed{H = \frac{p_\theta^2}{2m\left(l_0 - \alpha t\right)^2} - \frac{1}{2} m\alpha^2 - mg\left(l_0 - \alpha t\right) \cos\theta} \tag{1.5}$$

1.5 Hamilton's Equations of Motion

Find the Hamilton's equations of motion for θ and p_θ, and show that these equations of motion for θ and p_θ are equivalent to the equations of motion from subsection 1.2.

1.5 solution

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mg\left(l_0 - \alpha t\right) \sin\theta \quad \Rightarrow \quad \boxed{\dot{p}_\theta = -mg\left(l_0 - \alpha t\right) \sin\theta} \tag{1.6}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\left(l_0 - \alpha t\right)^2} \quad \Rightarrow \quad \boxed{\dot{\theta} = \frac{p_\theta}{m\left(l_0 - \alpha t\right)^2}} \tag{1.7}$$

$$\Rightarrow \quad \ddot{\theta} = \frac{\dot{p}_\theta}{m\left(l_0 - \alpha t\right)^2} - \alpha \frac{-2p_\theta}{m\left(l_0 - \alpha t\right)^3} = \frac{-mg\left(l_0 - \alpha t\right) \sin\theta}{m\left(l_0 - \alpha t\right)^2} + \alpha \frac{2m\left(l_0 - \alpha t\right)^2 \dot{\theta}}{m\left(l_0 - \alpha t\right)^3}$$

$$\Rightarrow \quad \left(l_0 - \alpha t\right) \ddot{\theta} = 2\alpha \dot{\theta} - g \sin\theta\,, \tag{1.8}$$

which is the same as equation 1.3 from subsection 1.2.

1.6 Energy vs Hamiltonian

In terms of m, α, l_0, and t, find the difference between the total energy, $T + U$, and the Hamiltonian for this system.

1.6 solution

$$E - H = \frac{p_\theta^2}{2m\left(l_0 - \alpha t\right)^2} + \frac{1}{2}m\alpha^2 - mg\left(l_0 - \alpha t\right)\cos\theta - \left[\frac{p_\theta^2}{2m\left(l_0 - \alpha t\right)^2} - \frac{1}{2}m\alpha^2 - mg\left(l_0 - \alpha t\right)\cos\theta\right]$$

$$\Rightarrow \quad \boxed{E - H = m\alpha^2} \tag{1.9}$$

So E and H only differ by a constant.