1 Uniform Gravitational Field

The idea of changing dynamical variables in order to split a Lagrangian into independent pieces can be applied to systems with noncentral-forces too.

Consider two particles, one with mass m_1 and position given by $\vec{r}_1$, and the other with mass m_2 and position given by $\vec{r}_2$, with both subject to a uniform constant gravitational field $\vec{g}$ and interacting with each other with potential energy $U_r(|\vec{r}_1 - \vec{r}_2|)$. **(a)** Write the Lagrangian for this system using coordinates $\vec{r}_1$ and $\vec{r}_2$, $L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$. **(b)** Make the change of variables from $\vec{r}_1$, $\vec{r}_2$ to $\vec{r}$, $\vec{R}$, where

$$\vec{r} = \vec{r}_1 - \vec{r}_2\,, \quad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\,, \tag{1.1}$$

and show that the transformed Lagrangian can be split into two parts like so

$$L = L_{\rm cm}(\vec{R}, \dot{\vec{R}}) + L_\mu(\vec{r}, \dot{\vec{r}})\,. \tag{1.2}$$

Identify $L_{\rm cm}$ and L_μ. You may want to define, μ, the reduced mass as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{1.3}$$

(c) Using equation 1.2, explain how the dynamics of $\vec{R}$ and $\vec{r}$ are independent of each other.

1.0 solution

$$L = T - U = \frac{1}{2}m_1\left(\dot{\vec{r}}_1\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{r}}_2\right)^2 - U_r(|\vec{r}_1 - \vec{r}_2|) - (-m_1\vec{g}\cdot\vec{r}_1) - (-m_2\vec{g}\cdot\vec{r}_2)$$

$$\Rightarrow \quad \textbf{(a)}\ \boxed{L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2) = \frac{1}{2}m_1\left(\dot{\vec{r}}_1\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{r}}_2\right)^2 - U_r(|\vec{r}_1 - \vec{r}_2|) + m_1\vec{g}\cdot\vec{r}_1 + m_2\vec{g}\cdot\vec{r}_2} \tag{1.4}$$

From inverting equations 1.1 and differentiating with respect to time we have

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}, \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}, \qquad \dot{\vec{r}}_1 = \dot{\vec{R}} + \frac{m_2}{m_1 + m_2}\dot{\vec{r}}, \quad \text{and} \quad \dot{\vec{r}}_2 = \dot{\vec{R}} - \frac{m_1}{m_1 + m_2}\dot{\vec{r}}. \tag{1.5}$$

Replacing $\vec{r}_1$, $\vec{r}_2$ with $\vec{R}$, $\vec{r}$ in the Lagrangian L gives

$$L = \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{m_1 + m_2}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{m_1 + m_2}\dot{\vec{r}}\right)^2 - U_r(|\vec{r}|) + m_1\vec{g}\cdot\left(\vec{R} + \frac{m_2}{m_1 + m_2}\vec{r}\right) + m_2\vec{g}\cdot\left(\vec{R} - \frac{m_1}{m_1 + m_2}\vec{r}\right)$$

$$\begin{aligned} = \; & \tfrac{1}{2}m_1\left(\dot{\vec{R}}\right)^2 + \tfrac{1}{2}\tfrac{m_1 m_2^2}{(m_1+m_2)^2}\left(\dot{\vec{r}}\right)^2 + \tfrac{m_1 m_2}{m_1+m_2}\left(\dot{\vec{R}}\cdot\dot{\vec{r}}\right) + \\ & \tfrac{1}{2}m_2\left(\dot{\vec{R}}\right)^2 + \tfrac{1}{2}\tfrac{m_1^2 m_2}{(m_1+m_2)^2}\left(\dot{\vec{r}}\right)^2 - \tfrac{m_1 m_2}{m_1+m_2}\left(\dot{\vec{R}}\cdot\dot{\vec{r}}\right) - \\ & U_r(|\vec{r}|) + m_1\vec{g}\cdot\vec{R} + m_2\vec{g}\cdot\vec{R} + \tfrac{m_1 m_2}{m_1+m_2}\vec{g}\cdot\vec{r} - \tfrac{m_1 m_2}{m_1+m_2}\vec{g}\cdot\vec{r} \end{aligned}$$

$$= \frac{1}{2}(m_1 + m_2)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\frac{m_1 m_2^2 + m_1^2 m_2}{(m_1 + m_2)^2}\left(\dot{\vec{r}}\right)^2 - U_r(|\vec{r}|) + (m_1 + m_2)\,\vec{g}\cdot\vec{R}\,. \tag{1.6}$$

Digression:

$$\frac{1}{2}\frac{m_1 m_2^2 + m_1^2 m_2}{(m_1 + m_2)^2}\left(\dot{\vec{r}}\right)^2 = \frac{1}{2}\frac{(m_1 + m_2)\,m_1 m_2}{(m_1 + m_2)^2}\left(\dot{\vec{r}}\right)^2 = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left(\dot{\vec{r}}\right)^2 = \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2\,, \tag{1.7}$$

where

$$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}\,.$$

So equation 1.6 becomes

$$L = \frac{1}{2}(m_1+m_2)\left(\dot{\vec{R}}\right)^2 + \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 - U_r(|\vec{r}|) + (m_1+m_2)\,\vec{g}\cdot\vec{R} \tag{1.8}$$

$$= \left[\frac{1}{2}(m_1+m_2)\left(\dot{\vec{R}}\right)^2 + (m_1+m_2)\,\vec{g}\cdot\vec{R}\right] + \left[\frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 - U_r(|\vec{r}|)\right]\,.$$

So if

$$L = L_{\rm cm}(\vec{R},\dot{\vec{R}}) + L_\mu(\vec{r},\dot{\vec{r}})\,, \tag{1.9}$$

then

$$L_{\rm cm}(\vec{R},\dot{\vec{R}}) = \frac{1}{2}(m_1+m_2)\left(\dot{\vec{R}}\right)^2 + (m_1+m_2)\,\vec{g}\cdot\vec{R}$$

and

$$L_\mu(\vec{r},\dot{\vec{r}}) = \frac{1}{2}\mu\left(\dot{\vec{r}}\right)^2 - U_r(|\vec{r}|) \tag{1.10}$$

(c) This is pretty obvious, but there goes. We can write Lagrange's equation for a given generalized coordinate q_i and a Lagrangian $L = L_1 + L_2$ as

$$\frac{\partial}{\partial q_i}(L_1+L_2) + \frac{\rm d}{{\rm d}t}\frac{\partial}{\partial \dot{q}_i}(L_1+L_2) = 0\,. \tag{1.11}$$

So if L_2 does not depend on q_i or $\dot{q}_i$ then equation 1.11 becomes

$$\frac{\partial}{\partial q_i}L_1 + \frac{\rm d}{{\rm d}t}\frac{\partial}{\partial \dot{q}_i}L_1 = 0\,, \tag{1.12}$$

and so L_2 does not have any effect on the dynamics of q_i. So in general the the dynamics of variables in a Lagrangian are not effected by terms in the Lagrangian that do not contain the variable. So in general if terms in a Lagrangian are independent of other terms in a the Lagrangian, and visa versa, then you have dynamical systems that are independent of each other. In our case the dynamics of $\vec{r}$ is independent of $\vec{R}$ and visa versa.

2 Orbit of a Free Particle

It was shown in your text, Thornton and Marion (equation 8.21), that the orbital path $r(\theta)$ can be found from the ordinary differential equation

$$\frac{{\rm d}^2}{{\rm d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)\,, \tag{2.1}$$

where l is the constant angular momentum, $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the two particles, and $F(r)$ is the force. Find the orbital path, $r(\theta)$, when there is no force, $F(r) = 0$. What is the common name of the curve of this orbital path?

2.0 solution

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=0 \quad\Rightarrow\quad \boxed{\frac{1}{r}=C_1\cos\theta+C_2\sin\theta}\,. \tag{2.2}$$

To more easily see what this polar plot, $r(\theta)$, is, we can look at it in Cartesian coordinates where $x = r\cos\theta$, and $y = r\sin\theta$. Writing equation 2.3 in terms of x, y, and r gives

$$\frac{1}{r}=C_1\frac{x}{r}+C_2\frac{y}{r} \quad\Rightarrow\quad \boxed{1=C_1x+C_2y}\,, \tag{2.3}$$

which is easy to identify as the equation of $\boxed{\text{a straight line}}$, as expected.

3 Find a Force from an Orbit

Find the central force, $F(r)$, that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where k is a constant.

3.0 solution

From equation 2.1

$$F(r)=-\frac{l^2}{\mu r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)-\frac{l^2}{\mu r^2}\frac{1}{r}=-\frac{l^2}{\mu r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-2\frac{1}{k\theta^3}\right)-\frac{l^2}{\mu r^2}\frac{1}{k\theta^2} \tag{3.1}$$

$$=-\frac{l^2}{\mu r^2}\left(6\frac{1}{k\theta^4}\right)-\frac{l^2}{\mu r^2}\frac{1}{k\theta^2}=-\frac{l^2}{\mu r^2}\left(6\frac{1}{k\left(\frac{r}{k}\right)^2}\right)-\frac{l^2}{\mu r^2}\frac{1}{k\frac{r}{k}} \tag{3.2}$$

$$\Rightarrow\quad \boxed{F(r)=-\frac{l^2}{\mu}\left(\frac{6k}{r^4}+\frac{1}{r^3}\right)}\,. \tag{3.3}$$

4 Time to Collide

Two particles are attracted to each other by gravity. They are in circular orbits about each other. The period of the orbital motion is τ. If the two particles are suddenly stopped in their orbits and allowed to be pulled straight toward each other by their gravitational attraction, show that they will collide after a time of $\frac{\tau}{4\sqrt{2}}$.

4.0 solution

For a circular orbit in polar coordinates (r,θ) $r = \text{const} = b$, and so is $\dot\theta$. We define $\dot\theta \equiv \omega$. Balancing force and centripetal acceleration for both particles as there go around the center of mass gives

$$m_1r_1\omega^2=G\frac{m_1m_2}{b^2} \quad\Rightarrow\quad r_1=G\frac{m_2}{\omega^2b^2}\,, \tag{4.1}$$

and

$$m_2\left(b-r_1\right)\omega^2=G\frac{m_1m_2}{b^2} \quad\Rightarrow\quad b-r_1=G\frac{m_1}{\omega^2b^2}\,, \tag{4.2}$$

where m_1 and m_2 are the masses of the two particles, b is the distance between the two particles, and r_1 is the distance from the center of mass to particle 1. Adding equations 4.1 and 4.2 gives

$$b = \frac{G}{\omega^2 b^2}(m_1 + m_2) \quad \Rightarrow \quad b = \frac{G m_1 m_2}{\mu \omega^2 b^2} \quad \Rightarrow \quad b^3 = \frac{G m_1 m_2}{\mu \omega^2} = \frac{G m_1 m_2 \tau^2}{4\pi^2 \mu}\,, \tag{4.3}$$

where we used $\omega\tau = 2\pi$ in the last step. We release the two particles from rest at a distance of b apart. The equation of motion for r can be written using conservation of energy with the initial r value of b giving

$$U_i + T_i = U_f + T_f \quad \Rightarrow \quad -\frac{G m_1 m_2}{b} = -\frac{G m_1 m_2}{r} + \frac{1}{2}\mu \dot{r}^2 \quad \Rightarrow \quad \mathrm{d}t = -\frac{\mu}{\sqrt{2 G m_1 m_2}} \frac{\mathrm{d}r}{\sqrt{\frac{1}{r} - \frac{1}{b}}}\,,$$

where we picked the minus sign since $\frac{\mathrm{d}r}{\mathrm{d}t}$ is less than zero when the particles move toward each other. Setting t to the time for colliding gives

$$\Rightarrow \quad t = -\frac{\mu}{\sqrt{2 G m_1 m_2}} \int_{r=b}^{0} \frac{\mathrm{d}r}{\sqrt{\frac{1}{r} - \frac{1}{b}}} = -\frac{\mu}{\sqrt{2 G m_1 m_2}} \int_{r=b}^{0} \frac{\mathrm{d}r}{\sqrt{\frac{b-r}{b\,r}}} = \frac{\mu\sqrt{b}}{\sqrt{2 G m_1 m_2}} \int_{r=0}^{b} \frac{\sqrt{r}\,\mathrm{d}r}{\sqrt{b-r}}\,. \tag{4.4}$$

With the change of variables $x^2 = r/b$, $2bx\,\mathrm{d}x = \mathrm{d}r$

$$t = \frac{\mu\sqrt{b}}{\sqrt{2 G m_1 m_2}} \int_{x=0}^{1} \frac{\sqrt{ax^2}\, 2bx\,\mathrm{d}x}{\sqrt{b - bx^2}} = \frac{\mu b\sqrt{2b}}{\sqrt{G m_1 m_2}} \int_{x=0}^{1} \frac{x^2 \mathrm{d}x}{\sqrt{1-x^2}}\,. \tag{4.5}$$

For the integral we have

$$\int_{x=0}^{1} \frac{x^2 \mathrm{d}x}{\sqrt{1-x^2}} = \left(-\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right)\Bigg|_{x=0}^{1} = \frac{\pi}{4}\,.$$

Combining this with equation 4.5 and 4.3 gives

$$t = \frac{\mu\sqrt{2} b^{\frac{3}{2}}}{\sqrt{G m_1 m_2}} \frac{\pi}{4} = \frac{\sqrt{2}\,\mu\sqrt{\frac{G m_1 m_2 \tau^2}{4\pi^2\mu}}}{\sqrt{G m_1 m_2}} \frac{\pi}{4} \quad \Rightarrow \quad \boxed{t = \frac{\tau}{4\sqrt{2}}}\,. \tag{4.6}$$