1 Uniform Gravitational Field

The idea of changing dynamical variables in order to split a Lagrangian into independent pieces can be applied to systems with noncentral-forces too.

Consider two particles, one with mass m_1 and position given by $\vec{r}_1$, and the other with mass m_2 and position given by $\vec{r}_2$, with both subject to a uniform constant gravitational field $\vec{g}$ and interacting with each other with potential energy $U_r(|\vec{r}_1 - \vec{r}_2|)$. **(a)** Write the Lagrangian for this system using coordinates $\vec{r}_1$ and $\vec{r}_2$, $L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$. **(b)** Make the change of variables from $\vec{r}_1$, $\vec{r}_2$ to $\vec{r}$, $\vec{R}$, where

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \,, \quad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \,, \tag{1.1}$$

and show that the transformed Lagrangian can be split into two parts like so

$$L = L_{\text{cm}}(\vec{R}, \dot{\vec{R}}) + L_\mu(\vec{r}, \dot{\vec{r}}) \,. \tag{1.2}$$

Identify L_{cm} and L_μ. You may want to define, μ, the reduced mass as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \,. \tag{1.3}$$

(c) Using equation 1.2, explain how the dynamics of $\vec{R}$ and $\vec{r}$ are independent of each other.

2 Orbit of a Free Particle

It was shown in your text, Thornton and Marion (equation 8.21), that the orbital path $r(\theta)$ can be found from the ordinary differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2} F(r) \,, \tag{2.1}$$

where l is the constant angular momentum, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the two particles, and $F(r)$ is the force. Find the orbital path, $r(\theta)$, when there is no force, $F(r) = 0$. What is the common name of the curve of this orbital path?

3 Find a Force from an Orbit

Find the central force, $F(r)$, that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where k is a constant.

4 Time to Collide

Two particles are attracted to each other by gravity. They are in circular orbits about each other. The period of the orbital motion is τ. If the two particles are suddenly stopped in their orbits and allowed to be pulled straight toward each other by their gravitational attraction, show that they will collide after a time of $\frac{\tau}{4\sqrt{2}}$.