1 $r(\theta)$ – Orbital Path for Another Force

A particle of mass m moves with angular momentum l and with total energy E about a fixed center with a force

$$F(r) = -\frac{k}{r^2} + \frac{\lambda}{r^3} \tag{1.1}$$

where k and λ are greater than zero, and r is the distance from the particle to the center. **(a)** Show that the equation for the orbit, $r(\theta)$, may have the form

$$\frac{\alpha}{r} = 1 + \epsilon\cos(\beta\theta)\,, \tag{1.2}$$

finding the constants α, ϵ, and β in terms of the given quantities m, k, λ, l, and E. Assume that the potential energies are zero at $r = \infty$ in defining the total energy, E. **(b)** For what values of β is the orbit closed?

1.0 solution

(a) I'll show all the detail here, but you will likely only need a subset of this by using the results of the Kepler problem or other short-cuts.

$$\mathrm{d}\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}r}\mathrm{d}r = \frac{\dot\theta}{\dot r}\mathrm{d}r\,. \tag{1.3}$$

$$U(r) = -\int\left(-\frac{k}{r^2} + \frac{\lambda}{r^3}\right)\mathrm{d}r = -\frac{k}{r} + \frac{\lambda}{2r^2}\,, \tag{1.4}$$

where we set $U(\infty) = 0$. From conservation of energy and angular momentum, $l = mr^2\dot\theta$,

$$E = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 - \frac{k}{r} + \frac{\lambda}{2r^2} = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\left(\frac{l^2}{m^2r^4}\right) - \frac{k}{r} + \frac{\lambda}{2r^2} = \frac{1}{2}m\dot r^2 + \frac{l^2}{2mr^2} - \frac{k}{r} + \frac{\lambda}{2r^2}$$

$$\Rightarrow \quad E = \frac{1}{2}m\dot r^2 + \left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)\frac{1}{r^2} - \frac{k}{r} \quad \Rightarrow \quad \dot r = \pm\sqrt{\frac{2}{m}\left[E - \left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)\frac{1}{r^2} - \frac{k}{r}\right]} \tag{1.5}$$

This with equation 1.3 gives

$$\Rightarrow \quad \mathrm{d}\theta = \pm\frac{\frac{l}{mr^2}\,\mathrm{d}r}{\sqrt{\frac{2}{m}\left[E - \left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)\frac{1}{r^2} + \frac{k}{r}\right]}} = \pm\frac{l}{\sqrt{2m}}\frac{\frac{\mathrm{d}r}{r^2}}{\sqrt{E - \left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)\frac{1}{r^2} + \frac{k}{r}}}\,. \tag{1.6}$$

With the change of variables $u = \frac{1}{r}$, $\frac{\mathrm{d}r}{r^2} = -\mathrm{d}u$, we get

$$\mathrm{d}\theta = \mp\frac{l}{\sqrt{2m}}\frac{\mathrm{d}u}{\sqrt{E + ku - \left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)u^2}}\,. \tag{1.7}$$

We complete the square of the term

$$\left(\frac{l^2}{2m} + \frac{\lambda}{2}\right)u^2 - ku\,,$$

giving

$$\mathrm{d}\theta = \mp\frac{l}{\sqrt{2m}}\frac{\mathrm{d}u}{\sqrt{\left(E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}\right) - \left(\sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}\,u - \frac{k}{\sqrt{\frac{2l^2}{m} + 2\lambda}}\right)^2}}\,. \tag{1.8}$$

Change variables to

$$y = \sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}\, u - \frac{k}{\sqrt{\frac{2l^2}{m} + 2\lambda}} \quad \text{so} \quad \mathrm{d}y = \sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}\, \mathrm{d}u \tag{1.9}$$

$$\Rightarrow \quad \theta = \mp \frac{1}{\sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}} \frac{l}{\sqrt{2m}} \int \frac{\mathrm{d}y}{\sqrt{\left[\left(E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}\right) - y^2\right]}} = \mp \frac{1}{\sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}} \frac{l}{\sqrt{2m}} \cos^{-1}\left(\frac{y}{\sqrt{E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}}}\right), \tag{1.10}$$

where we have set the integration constant to zero, so that $\theta = 0$ will have a minimum r value.

$$\Rightarrow \quad \cos\left(\sqrt{\frac{1}{2m} + \frac{\lambda}{2l^2}}\sqrt{2m}\,\theta\right) = \frac{\sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}\, u - \frac{k}{\sqrt{\frac{2l^2}{m} + 2\lambda}}}{\sqrt{E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}}}.$$

$$\Rightarrow \quad \frac{k}{\sqrt{\frac{2l^2}{m} + 2\lambda}} + \sqrt{E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}} \cos\left(\sqrt{1 + \frac{m\lambda}{l^2}}\,\theta\right) = \sqrt{\frac{l^2}{2m} + \frac{\lambda}{2}}\, u\,,$$

$$\Rightarrow \quad \frac{\frac{l^2 + m\lambda}{mk}}{r} = 1 + \frac{\sqrt{\frac{2l^2}{m} + 2\lambda}}{k} \sqrt{E + \frac{k^2}{2\frac{l^2}{m} + 2\lambda}} \cos\left(\sqrt{1 + \frac{m\lambda}{l^2}}\,\theta\right)$$

$$\Rightarrow \quad \boxed{\frac{\frac{l^2 + m\lambda}{mk}}{r} = 1 + \left[\sqrt{\frac{2E\left(l^2 + m\lambda\right)}{k^2}} + 1\right] \cos\left(\sqrt{1 + \frac{m\lambda}{l^2}}\,\theta\right)}. \tag{1.11}$$

So

$$\boxed{\alpha \equiv \frac{l^2 + m\lambda}{mk}}, \quad \boxed{\beta \equiv \sqrt{1 + \frac{m\lambda}{l^2}}} \quad \text{and} \quad \boxed{\epsilon \equiv \sqrt{\frac{2E\left(l^2 + m\lambda\right)}{k^2} + 1}}, \tag{1.12}$$

and

$$\frac{\alpha}{r} = 1 + \epsilon \cos(\beta\theta) \tag{1.13}$$

(b) In order for there to be a closed orbit the phase in cosine term , $\beta\theta$, must be $n2\pi$ when $\theta = m2\pi$ where m and n are integers, and the orbit will close after m revolutions. Therefore, in order for there to be a closed orbit

$$\beta\left(m2\pi\right) = n2\pi \quad \Rightarrow \quad \beta = \frac{n}{m} = \text{rational number} \quad \Rightarrow \quad \sqrt{1 + \frac{m\lambda}{l^2}} = \text{rational number}\,. \tag{1.14}$$