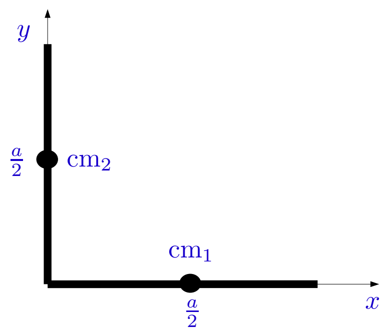
1 Center of Mass of a Two Wires

Two identical uniform straight wires, both with length a, are in the x-y plane. One wire has one end at the origin and goes along the positive x axis. The other wire has one end at the origin and goes along the positive y axis. Find the center of mass of the two wires.

1.0 solution



Let the mass of each wire be M.

$$\vec{R} = \frac{\sum\limits_{\alpha} m_\alpha \vec{r}_\alpha}{\sum\limits_{\alpha} m_\alpha} = \frac{m_1 \vec{r}_{\mathrm{cm}\,1} + m_2 \vec{r}_{\mathrm{cm}\,2}}{M + M} = \frac{1}{2M}\left[\left(M\,\frac{a}{2}\hat{x} + 0\,\hat{y}\right) + \left(0\,\hat{x} + M\,\frac{a}{2}\hat{y}\right)\right]$$

$$\Rightarrow \quad \boxed{\vec{R} = \frac{a}{4}\hat{x} + \frac{a}{4}\hat{y}}. \tag{1.1}$$

2 Center of Mass of a Cone

A uniformly solid cone has a base diameter $2a$ and height h. The base of the cone is at the origin. The cone points along the z direction. Find the center of mass of this cone.

2.0 solution

From symmetry the x and y components of the center of mass are zero, therefore

$$\vec{R} = z_{\mathrm{cm}}\,\hat{z}\,, \tag{2.1}$$

where z_{cm} is the z-component of the center of mass which we must find from an integration, $\hat{x}$ is a unit vector in the x-direction, and $\hat{y}$ is a unit vector in the y-direction. We find z_{cm} by summing up vertical disks of height $\mathrm{d}z$, radius $r = a - \frac{a}{h}\,z$, and mass density $\rho = \frac{M}{\frac{1}{3}\pi\,a^2 h}$ where M is the total mass of the cone.

$$z_{\mathrm{cm}} = \frac{1}{M}\int \rho\, z\, \mathrm{d}\,(\mathrm{Volume}) = \frac{1}{M}\int_{z=0}^{h}\left(\frac{M}{\frac{1}{3}\pi\,a^2 h}\right) z \left[\pi\left(a - \frac{a}{h}\,z\right)^2 \mathrm{d}z\right]$$

$$= \frac{3}{h}\int_{z=0}^{h}\left(z - \frac{2}{h}\,z^2 + \frac{1}{h^2}\,z^3\right)\mathrm{d}z = \frac{3}{h}\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)h^2 = 3\,\frac{6-8+3}{12}h$$

$$\Rightarrow \quad \boxed{z_{\mathrm{cm}} = \frac{h}{4}} \quad \Rightarrow \quad \boxed{\vec{R} = \frac{h}{4}\hat{z}}. \tag{2.2}$$

3 Center of Mass of a Circular Arc

A uniform wire subtends a circular arc with angular length θ and radius a. The wire is in the x-y plane. The arc is a portion of a circle that is centered at the origin with the y position of the center of mass being zero and the center of mass having a positive x position. Find the x component of the position of the center of mass of this wire.

3.0 solution

From symmetry the y and z components of the center of mass are zero, therefore

$$\vec{R} = x_{\rm cm}\,\hat{x} \tag{3.1}$$

where $x_{\rm cm}$ is the x-component of the center of mass which we must find from an integration, and $\hat{x}$ is a unit vector in the x-direction. Let M be the total mass of the wire and $\lambda = \frac{M}{a\,\theta}$ is the linear mass density.

$$x_{\rm cm} = \frac{1}{M}\int x\,\lambda\,{\rm d}s = \frac{1}{M}\int x\,\frac{M}{a\,\theta}\,{\rm d}s = \frac{1}{a\,\theta}\int x\,{\rm d}s \tag{3.2}$$

where ${\rm d}s$ is the length of a small piece of the wire. We can put things in terms of the angle, θ', measured from the x axis, giving

$${\rm d}s = a\,{\rm d}\theta' \quad \text{and} \quad x = a\,\cos\theta'\,. \tag{3.3}$$

So

$$x_{\rm cm} = \frac{1}{a\,\theta}\int a\,\cos\theta'\,a\,{\rm d}\theta' = \frac{a}{\theta}\int_{\theta'=-\frac{\theta}{2}}^{\frac{\theta}{2}}\cos\theta'\,{\rm d}\theta' = \frac{2a}{\theta}\,\sin\frac{\theta}{2} \quad \Rightarrow \quad \boxed{\vec{R} = \frac{2a}{\theta}\,\sin\frac{\theta}{2}\,\hat{x}}\,. \tag{3.4}$$

4 Center of Gravity

The center of gravity of a system of particles is the point about which external gravitational force exerts no net torque. For a uniform gravitational field, show that the center of gravity is identical to the center of mass for a system of particles.

4.0 solution

Let $\vec{r}_\alpha$ be the position of one of the particles, m_α be the mass of one of the particles, and $\vec{g}$ be the constant uniform gravitational field. The condition that there is no net torque from a uniform gravitational field on a system of particles, $N_{\rm net}$, can be written as

$$N_{\rm net} = \sum_\alpha\left[(\vec{r}_\alpha - \vec{r}) \times \vec{F}_g\right] = \sum_\alpha\left[(\vec{r}_\alpha - \vec{r}) \times (m_\alpha\vec{g})\right] \tag{4.1}$$

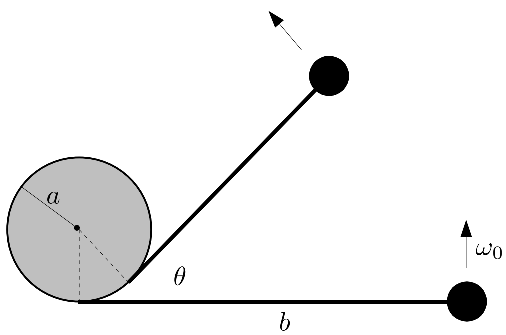
where $\vec{F}_g = m_\alpha\vec{g}$ is the force of the uniform gravitational field $\vec{g}$ on the α-th particle, and $\vec{r}$ is an arbitrary position which we are computing the torque about. This gives

$$N_{\rm net} = \sum_\alpha m_\alpha\left[(\vec{r}_\alpha - \vec{r}) \times \vec{g}\right] = \left[\sum_\alpha m_\alpha\,(\vec{r}_\alpha - \vec{r})\right] \times \vec{g} = \left(\sum_\alpha m_\alpha\vec{r}_\alpha - \sum_\alpha m_\alpha\vec{r}\right) \times \vec{g} = \left(M\vec{R} - M\vec{r}\right) \times \vec{g}$$

$$\Rightarrow \quad N_{\text{net}} = M\left(\vec{R} - \vec{r}\right) \times \vec{g}\,, \tag{4.2}$$

where $M \equiv \sum_\alpha m_\alpha$ is the total mass in the system, and $\vec{R} \equiv \frac{\left(\sum_\alpha m_\alpha \vec{r}_\alpha\right)}{M}$ is the center of mass position of the system of particles. So N_{net} will be zero if the center of mass position is equal to the position where we measure the torque about from a uniform gravitational field, no matter what is the direction of the gravitational field, $\vec{g}$, is. So the center of gravity must be identical to the center of mass for the system.

5 Wrapping a String with Bob Around a Pole



A particle of mass m at the end of a light stretch-less string wraps itself about a fixed vertical cylinder of radius a. All the motion is in the horizontal plane (no gravity). The initial angular velocity of the string is ω_0 when the distance from the particle to the point of contact of the string and cylinder is b. Two positions of the particle and string are shown above. Find the angular velocity ω and the tension, τ, after the cord has turned through an additional angle θ. Hint: Energy is conserved.

5.0 solution

From conservation of energy we can write

$$T_1 = T_2 \quad \Rightarrow \quad \frac{1}{2}mr_1^2\omega_0^2 = \frac{1}{2}mr_2^2\omega^2 \quad \Rightarrow \quad \frac{1}{2}mb^2\omega_0^2 = \frac{1}{2}mr_2^2\omega^2 \tag{5.1}$$

where r_2 is the length of the string. From the geometry

$$r_2 = b - a\theta\,. \tag{5.2}$$

so from this and equation 5.1 we get

$$\frac{1}{2}mb^2\omega_0^2 = \frac{1}{2}m\left(b - a\theta\right)^2\omega^2 \quad \Rightarrow \quad \boxed{\omega = \frac{b\,\omega_0}{b - a\theta}}. \tag{5.3}$$

The tension is given by Newton's 2nd law applied along the direction of the string given the centripetal acceleration

$$\tau = m\omega^2\left(b - a\theta\right) = m\left(\frac{b}{b - a\theta}\right)^2\omega_0^2\left(b - a\theta\right) \quad \Rightarrow \quad \boxed{\tau = \frac{mb^2\omega_0^2}{b - a\theta}}. \tag{5.4}$$

One might ask: How do we know that energy is conserved? The tension in the string is always perpendicular to the velocity of the particle, therefore the force from the string can do no work on the particle, and so the kinetic energy in the particle is constant, and the energy is the kinetic energy in the particle which is constant, so the energy is conserved.

6 Rope Sliding off a Table

A massive stretch-less flexible rope of length 1.0 m slides without friction on a flat table top. The rope is initially released from rest with 30 cm of rope hanging straight down over the edge of the table. Find the time, τ, from when the rope is released, when the last part of the rope will leave the table top. 15 points extra HW credit for figuring out what's wrong with this problem and solving it correctly (As explained in class).

6.0 solution

We'll assume that energy is conserved, the rope moves along the table, and then straight down when when it comes off the table. Though this is not correct it will at least give a lower bound to the value of τ.

The total energy in the rope, E, is the sum of the kinetic plus the gravitational potential energies. We can set the gravitational potential energy of the rope that is horizontally along the top of the table as zero, so the only gravitational potential energy will be from the rope that is hanging off the table, and it will be less than zero. Let M be the total mass of the rope. Let $L = 1.0\text{m}$ be the total length of the rope. Let x be the length of the rope hanging down off the table, and so the length of rope on the table is $L - x$. The linear mass density of the rope is $\lambda = \frac{M}{L}$.

$$E = \frac{1}{2}M\dot{x}^2 - (\lambda x)\, g x_{\text{cm}} = \frac{1}{2}M\dot{x}^2 - \frac{M}{L}x\, g\frac{x}{2}\,. = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}\frac{M}{L}g\, x^2\,. \tag{6.1}$$

With the rope initially at rest, with length $x_0 = 30cm$ of rope hanging off the table we have

$$-\frac{1}{2}\frac{M}{L}g\, x_0^2 = \frac{1}{2}M\dot{x}^2 - \frac{1}{2}\frac{M}{L}g\, x^2 \quad \Rightarrow \quad -\frac{g}{L}x_0^2 = \dot{x}^2 - \frac{g}{L}x^2 \quad \Rightarrow \quad \dot{x} = \sqrt{\frac{g}{L}\left(x^2 - x_0^2\right)}$$

$$\Rightarrow \quad \tau = \sqrt{\frac{L}{g}}\int_{x=x_0}^{L} \frac{\mathrm{d}x}{\sqrt{x^2 - x_0^2}} \quad \Rightarrow \quad \tau = \sqrt{\frac{L}{g}}\cosh^{-1}\frac{x}{x_0} \approx \sqrt{\frac{1\text{m}}{9.8\frac{\text{m}}{\text{s}^2}}}\cosh^{-1}\frac{1}{0.3} \quad \Rightarrow \quad \boxed{\tau \approx 0.599\text{s}}\,. \tag{6.2}$$