In this homework, you may use results from the text. Put your answers in terms of given quantities. Though the calculations here in can be done as plug-ins, it's highly recommended that you do some deriving on your own to gain some understanding.

# 1 Scattering from Hard Spheres

A target consists of $N_{\rm tar}$ sparsely distributed smooth hard spheres of mass $m_2$. The radius of each these target spheres is $R$. A beam of small particles with mass $m_1$ is incident on the target, where $m_1 \ll m_2$. The speed of the incident particles is $u_1$. The cross-sectional area of the entire target is $A$. $A \gg R^2$. The beam of small particles shoots a total of $N_{\rm inc}$ particles at the target, where $N_{\rm inc}$ is a large number. The cross-sectional area of the beam of small particles is much larger than $R^2$, but smaller than $A$. The small particles scatter elastically from the hard spheres.

## 1.1 Calculations

**(a)** Out of all the incident particles, $N_{\rm inc}$, what is the number of $m_1$ particles that are not scattered by any hard spheres (the number of $m_1$ particles that go through the target with no deflection), $N_{\rm un}$. **(b)** What is number of particles per unit area that go off at an angle of $\theta$ measured from the direction of the incident beam, where $\theta = 0$ would be no scattering (deflection), and at a distance of $D$ from the target, where $D^2 \gg A$.

**1.1 solution**

**(a)** The number of particles that are not scattered will be equal to the fraction of the target area that is not covered by targets times the total number of incident $m_1$ particles:

$$(\mathbf{a})\ \boxed{N_{\rm un} = N_{\rm inc}\frac{A - N_{\rm tar}\pi R^2}{A} = N_{\rm inc}\left(1 - \frac{N_{\rm tar}\pi R^2}{A}\right)}\,. \tag{1.1}$$

**(b)** Let $n_{\rm sc}$ be the number of particles per unit area that go off at an angle of $\theta$, and $N_{\rm sc}$ be the number of particles that go off at an angle of $\theta$. In a differential sized region of solid angle we have the small number of scatterers $\mathrm{d}N_{\rm sc}$ given by

$$\mathrm{d}N_{\rm sc} = N_{\rm inc}\frac{N_{\rm tar}}{A}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega \tag{1.2}$$

where $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ is the differential scattering cross-section ($\sigma$ in Thorton) for a single scatterer, and $\mathrm{d}\Omega$ is a small region of solid angle. For a hard sphere scatterer

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4}\,. \tag{1.3}$$

In our case we can measure the solid angle as

$$\mathrm{d}\Omega = \frac{\mathrm{d}a}{D^2} \tag{1.4}$$

where $\mathrm{d}a$ is a differential area, and we used the radius $D$ for the sphere to measure the solid angle on. Putting things together we get

$$\mathrm{d}N_{\rm sc} = N_{\rm inc}\frac{N_{\rm tar}}{A}\frac{R^2}{4}\frac{\mathrm{d}a}{D^2} \quad \Rightarrow \quad (\mathbf{b})\ \boxed{\frac{\mathrm{d}N_{\rm sc}}{\mathrm{d}a} = N_{\rm inc}\frac{N_{\rm tar}\,R^2}{4\,A\,D^2}} \tag{1.5}$$

Checking: By integrating $\frac{\mathrm{d}N_{\rm sc}}{\mathrm{d}a}$ times the area over a sphere of radius $D$ we get

$$N_{\rm sc} = N_{\rm inc}\frac{N_{\rm tar}\,R^2}{4\,A\,D^2}\left(4\pi D^2\right) = N_{\rm inc}\frac{N_{\rm tar}\pi\,R^2}{A} \tag{1.6}$$

as expected. So

$$N_{\rm inc} = N_{\rm sc} + N_{\rm un} = N_{\rm inc}\frac{N_{\rm tar}\pi R^2}{A} + N_{\rm inc}\left(1 - \frac{N_{\rm tar}\pi R^2}{A}\right) , \tag{1.7}$$

the sum of the scattered particles plus unscattered particles is equal to the total number of incident particles.

## 1.2 Conceptual Questions

This is an idealized scattering experiment. We have been a little more explicit in describing the relative parameter sizes than what is typically found in text book problems. There are a few approximations used in coming up with our simple scattering model. Without these simplifying assumptions our simple scattering model would not be the same.

Give **brief** answers to the following questions. Think about what would happen to our simple model in extreme cases. How would our scattering model be different or invalid if:

1. we had more hard spheres, such that they become densely distributed in the target,

   It would be likely to have more than one collision per beam particle, and our model does not cover that.

2. the beam of $m_1$ particles was not sparse, but dense in particles (high intensity),

   The beam particle may interact with each other.

3. $N_{\rm inc}$ was not a large number,

   We can still calculate the probabilities of scattering in a given region.

4. the cross-sectional area of the beam of small particles was not smaller than $A$,

   The part of the beam that is not incident on the target will not have any particles scattered.

5. the cross-sectional area of the beam of small particles was smaller than $R^2$,

   You will not get the distribution of scattering trajectories that our model predicts. You may get no scattering at all if beam is not pointing at a target particle.

6. the beam of small particles was a beam of larger particles of radius $r$ which is close in size to $R$, the radius of the target spheres,

   This will increase the total scatter cross-section, and the number of particles scattered.

7. we were given the intensity of the particle beam in particles per unit area per unit time, instead of just the total number of particles,

We could calculate the particles per unit area per unit time, that are scattered, by replacing the number of particles in the beam with the intensity of the particle beam.

8. the target spheres were not smooth when compared with the size of the beam particles,

The differential scattering cross-section will be different.

9. we did not have $m_1 \ll m_2$, or

The Lab frame of reference will not give the same results as the Center-of-mass frame.

10. the hard spheres were distributed in a pattern.

Shouldn't make a difference. We're using particles not waves. So classically there will be not interference effects.

11. FYI: Here's one we forgot: What happens when $D^2$ is not much greater than $A$ and the width of the beam.

Possible answer: The angle of deflection measured will depend on the position of the scattering target particle that scattered it, were as with $D^2 \gg A$ the position of the scattering target particle could be considered to be the same for all scattering target particles.

Remember be **brief**. Just show that you know what the effect of the change is.

## 2 Rutherford Scattering

Given the same experiment as in problem 1, except with the beam particles now interacting with the target particles with a repulsive force given by $\frac{k}{r^2}$, where $k$ is a constant, $k > 0$, $r$ is the distance between a target particle and a beam particle, and now the radius of the target particles, $R$, is negligible. So we still have $N_{\mathrm{tar}}$, $m_1$, $m_2$, $u_1$, $A$, and $N_{\mathrm{inc}}$ as defined in problem 1.

### 2.1 Particles Deflected at Angle $\theta$

What is number of particles per unit area that go off at a deflection angle of $\theta$ measured from the direction of the incident beam, where $\theta = 0$ would be no scattering, and at a distance of $D$ from the target, where $D^2 \gg A$.

**2.1 solution**

Let $n_{\mathrm{sc}}$ be the number of particles per unit area that go off at an angle of $\theta$, and $N_{\mathrm{sc}}$ be the number of particles that go off at an angle of $\theta$. In a differential sized region of solid angle we have the small number of scatterers $\mathrm{d}N_{\mathrm{sc}}$ given by

$$\mathrm{d}N_{\mathrm{sc}} = N_{\mathrm{inc}} \frac{N_{\mathrm{tar}}}{A} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrm{d}\Omega \tag{2.1}$$

where $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ is the differential scattering cross-section ($\sigma$ in Thorton) for a single scatterer, and $\mathrm{d}\Omega$ is a small region of solid angle. For Rutherford scattering

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{k}{2m_1 u_1^2 \sin^2\frac{\theta}{2}}\right)^2 . \tag{2.2}$$

In our case we can measure the solid angle as

$$\mathrm{d}\Omega = \frac{\mathrm{d}a}{D^2} \tag{2.3}$$

where $\mathrm{d}a$ is a differential area, and we used the radius $D$ for the sphere to measure the solid angle on. Putting things together we get

$$\mathrm{d}N_{\mathrm{sc}} = N_{\mathrm{inc}}\frac{N_{\mathrm{tar}}}{A}\left(\frac{k}{2m_1 u_1^2 \sin^2\frac{\theta}{2}}\right)^2 \frac{\mathrm{d}a}{D^2} \quad\Rightarrow\quad \boxed{\frac{\mathrm{d}N_{\mathrm{sc}}}{\mathrm{d}a} = N_{\mathrm{inc}}\frac{N_{\mathrm{tar}}}{A}\left(\frac{k}{2m_1 u_1^2\, D \sin^2\frac{\theta}{2}}\right)^2}. \tag{2.4}$$

## 2.2 Too Small Deflection Angle

In this case, with Rutherford scattering, the incident particles are interacting with the target particles over an infinitely long distance. At about what deflection angle is large enough such that we can consider the deflected particle to be scattered from just one target particle. Call this small deflection angle $\theta_1$. For this problem ignore electron screening effects, though for a real physical system that is very important. We never said we had electrons (or anything physically real) in this problem.

Hint: Find the angle of deflection when the impact parameter is about half the size of the inter-target-particle spacing. If the impact parameter of a beam particle is about half the size of inter-target-particle spacing then it is probable that we will have a small scattering that is due to interactions with more than one particle. So we see that when the deflection angle is too small our two particle interaction model of scattering can no longer be used to calculate the deflection angle.

**2.2 solution**

From Taylor equation 14.31, the impact parameter $b$ is

$$b = \frac{k}{m_1 u_1^2}\cot\frac{\theta}{2} . \tag{2.5}$$

The average inter-target-particle spacing is $\sqrt{\frac{A}{N_{\mathrm{tar}}}}$, so with $b = \frac{1}{2}\sqrt{\frac{A}{N_{\mathrm{tar}}}}$ we can solve for $\theta = \theta_1$ giving

$$\cot\frac{\theta_1}{2} = \frac{m_1 u_1^2}{k}\,\frac{1}{2}\sqrt{\frac{A}{N_{\mathrm{tar}}}} \quad\Rightarrow\quad \tan\frac{\theta_1}{2} = \frac{2\,k}{m_1 u_1^2}\sqrt{\frac{N_{\mathrm{tar}}}{A}} \quad\Rightarrow\quad \boxed{\theta_1 = 2\tan^{-1}\left(\frac{2\,k}{m_1 u_1^2}\sqrt{\frac{N_{\mathrm{tar}}}{A}}\right)}. \tag{2.6}$$

So if you want resolve scattering angles smaller than this $\theta_1$ you'd better use a different scattering theory.

This shows that if we increase $N_{\mathrm{tar}}$ we must look at larger deflection angles to measure results that agree with our theory, which makes sense. Increasing beam energy, $m_1 u_1^2$, makes the angle get smaller, which makes sense too.