1 Scattering from $\frac{k}{r^3}$ Force

A fixed force center elastically scatters a particle of mass m according to the central force $F(r) = k/r^3$. If the initial velocity of the particle is u_0, show that the differential scattering cross section is

$$\sigma(\theta) = \frac{k\,\pi^2(\pi-\theta)}{mu_0^2\,\theta^2(2\pi-\theta)^2\sin\theta}\,. \tag{1.1}$$

Note: This is using Thorton's notion. In comparing Thorton and Taylor's notion we have (Thorton = Taylor):

$$\sigma(\theta) = \frac{\mathrm{d}\sigma(\theta)}{\mathrm{d}\Omega}\,. \tag{1.2}$$

Hint: Start with equations **9.122** and **9.123** from Thorton and use the change of variables $u = 1/r$.

1.0 solution

We start by combining equations **9.122** and **9.123** from Thorton which gives

$$\frac{1}{2}(\pi-\theta) = \int_{r=r_{\min}}^{\infty} \frac{\left(\frac{b}{r^2}\right)\mathrm{d}r}{\sqrt{1-\left(\frac{b^2}{r^2}\right)-\left(\frac{U}{T_0}\right)}} \tag{1.3}$$

where we have set $T_0 = T_0'$ because the center-of-mass frame and the Lab frame are the same frame.

$$U(r) = -\int \frac{k}{r^3}\,\mathrm{d}r = \frac{1}{2}\frac{k}{r^2} \quad\Rightarrow\quad \frac{1}{2}(\pi-\theta) = \int_{r=r_{\min}}^{\infty} \frac{\left(\frac{b}{r^2}\right)\mathrm{d}r}{\sqrt{1-\frac{b^2}{r^2}-\frac{k}{2\,T_0\,r^2}}} = \int_{r=r_{\min}}^{\infty} \frac{\left(\frac{b}{r^2}\right)\mathrm{d}r}{\sqrt{1-\left(b^2+\frac{k}{2\,T_0}\right)\frac{1}{r^2}}}\,. \tag{1.4}$$

We substitute $u = b/r$, with $\mathrm{d}u = -\frac{b}{r^2}\,\mathrm{d}r$ giving

$$\frac{1}{2}(\pi-\theta) = \int_{u=0}^{\frac{b}{r_{\min}}} \frac{\mathrm{d}u}{\sqrt{1-\left(1+\frac{k}{2\,T_0\,b^2}\right)u^2}} = \frac{1}{\sqrt{1+\frac{k}{2\,T_0\,b^2}}}\int_{u=0}^{\frac{b}{r_{\min}}} \frac{\mathrm{d}u}{\sqrt{\frac{1}{1+\frac{k}{2\,T_0\,b^2}}-u^2}}$$

$$\Rightarrow\quad \frac{1}{2}(\pi-\theta) = \frac{1}{\sqrt{1+\frac{k}{2\,T_0\,b^2}}}\,\sin^{-1}\left(\sqrt{1+\frac{k}{2\,T_0\,b^2}}\,u\right)\Bigg|_{u=0}^{\frac{b}{r_{\min}}} = \frac{1}{\sqrt{1+\frac{k}{2\,T_0\,b^2}}}\,\sin^{-1}\left(\sqrt{1+\frac{k}{2\,T_0\,b^2}}\,\frac{b}{r_{\min}}\right)\,. \tag{1.5}$$

We can find $r_{\min}$ using the conservation of energy, with the total energy being $\frac{1}{2}mu_0^2$, and by setting $r = r_{\min}$ and $\dot{r} = 0$, like so

$$\frac{1}{2}mu_0^2 = \frac{l^2}{2mr_{\min}^2} + \frac{k}{2r_{\min}^2} \quad\Rightarrow\quad mu_0^2 = \frac{m^2b^2u_0^2}{mr_{\min}^2} + \frac{k}{r_{\min}^2} = \left(mb^2u_0 + k\right)\frac{1}{r_{\min}^2} \quad\Rightarrow\quad r_{\min} = \sqrt{\frac{mb^2u_0+k}{mu_0^2}}\,. \tag{1.6}$$

Plugging this into equation 1.5 and setting $T_0 = \frac{1}{2}mu_0^2$ gives

$$\frac{1}{2}(\pi-\theta) = \frac{1}{\sqrt{1+\frac{k}{mu_0^2\,b^2}}}\,\sin^{-1}\left(\sqrt{1+\frac{k}{mu_0^2\,b^2}}\,b\sqrt{\frac{mu_0^2}{mb^2u_0^2+k}}\right) \quad\Rightarrow\quad \frac{1}{2}(\pi-\theta) = \frac{1}{\sqrt{1+\frac{k}{mu_0^2\,b^2}}}\,\frac{\pi}{2}\,. \tag{1.7}$$

Solving for $b(\theta)$ we get

$$1+\frac{k}{mu_0^2\,b^2} = \frac{\pi^2}{(\pi-\theta)^2} \quad\Rightarrow\quad \frac{k}{mu_0^2\,b^2} = \frac{\pi^2}{(\pi-\theta)^2} - 1 \quad\Rightarrow\quad \frac{k}{mu_0^2\,b^2} = \frac{\pi^2-(\pi-\theta)^2}{(\pi-\theta)^2}$$

$$\Rightarrow \quad \frac{k}{m u_0^2 b^2} = \frac{\pi^2 - (\pi-\theta)^2}{(\pi-\theta)^2} \quad \Rightarrow \quad \frac{m u_0^2 b^2}{k} = \frac{(\pi-\theta)^2}{\pi^2 - (\pi-\theta)^2} = \frac{(\pi-\theta)^2}{\theta\,(2\pi-\theta)} \quad \Rightarrow \quad b(\theta) = \sqrt{\frac{k}{m u_0^2}}\,\frac{\pi-\theta}{\sqrt{\theta\,(2\pi-\theta)}}\,. \tag{1.8}$$

So with this and Thorton equation **9.120** we get

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{b}{\sin\theta}\sqrt{\frac{k}{m u_0^2}}\left|\frac{-\sqrt{\theta\,(2\pi-\theta)} - \frac{\frac{1}{2}\,(\pi-\theta)}{\sqrt{\theta(2\pi-\theta)}}\,(2\pi-2\theta)}{\theta\,(2\pi-\theta)}\right|$$

$$= \frac{b}{\sin\theta}\sqrt{\frac{k}{m u_0^2}}\,\frac{\left|\frac{-\theta(2\pi-\theta)-(\pi-\theta)^2}{\sqrt{\theta(2\pi-\theta)}}\right|}{\theta\,(2\pi-\theta)} = \frac{b}{\sin\theta}\sqrt{\frac{k}{m u_0^2}}\,\frac{\left|\frac{-2\pi\theta+\theta^2-\pi^2+2\pi\theta-\theta^2}{\sqrt{\theta(2\pi-\theta)}}\right|}{\theta\,(2\pi-\theta)}$$

$$= \frac{b}{\sin\theta}\sqrt{\frac{k}{m u_0^2}}\,\frac{\pi^2}{\left[\theta\,(2\pi-\theta)\right]^{\frac{3}{2}}}\,.$$

Plugging in the $b(\theta)$ gives

$$\sigma(\theta) = \left[\sqrt{\frac{k}{m u_0^2}}\,\frac{\pi-\theta}{\sqrt{\theta\,(2\pi-\theta)}}\right]\frac{1}{\sin\theta}\sqrt{\frac{k}{m u_0^2}}\,\frac{\pi^2}{\left[\theta\,(2\pi-\theta)\right]^{\frac{3}{2}}} = \frac{k\,\pi^2\,(\pi-\theta)}{m u_0^2\,\theta^2\,(2\pi-\theta)^2\sin\theta}$$

$$\Rightarrow \quad \boxed{\sigma(\theta) = \frac{k\,\pi^2\,(\pi-\theta)}{m u_0^2\,\theta^2\,(2\pi-\theta)^2\sin\theta}}\,. \tag{1.9}$$