# 1 Minimum Exhaust Speed

A rocket has an initial mass of $m$ and a constant fuel burn rate of $\alpha$. The acceleration do to gravity is $g$. What is the minimum exhaust speed that will allow the rocket to lift off immediately after firing?

**1.0 solution**

The rocket equation with the force from a uniform gravitation field with magnitude $g$ is

$$-mg = m\dot{v} + u\dot{m} \tag{1.1}$$

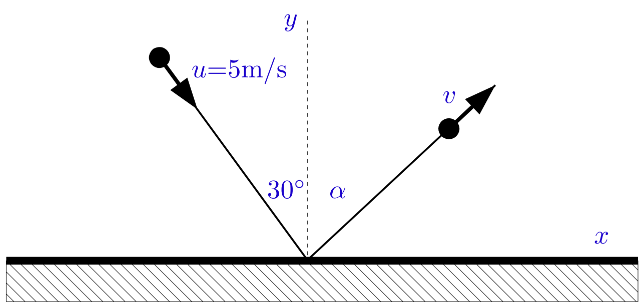
where $m$ is the mass of the rocket, $u$ is the exhaust speed, and $v$ is the speed of the rocket. When we have the minimum exhaust speed $u_{\min}$, $\dot{v}$ will be zero, so

$$-mg = 0 + u_{\min}(-\alpha) \quad \Rightarrow \quad \boxed{u_{\min} = \frac{mg}{\alpha}}. \tag{1.2}$$

# 2 Bouncing a Ball

A steel ball strikes a smooth heavy steel plate at an angle of 30° from the normal, and with speed of $u = 5$ m/s. The coefficient of restitution is 0.8. At what angle, $\alpha$, and speed, $v$, does the steel ball bounce off the plate with?

**2.0 solution**




Because we have a smooth plate momentum is conserved in the $x$ direction, so we have

$$u \sin 30° = v \sin\alpha\,. \tag{2.1}$$

From the definition of the coefficient of restitution, $\epsilon = 0.8$, we have the speeds along the $y$ direction related like

$$\epsilon\, u \cos 30° = v \cos\alpha\,. \tag{2.2}$$

Dividing equation 2.1 by equation 2.2 gives

$$\frac{\tan 30°}{\epsilon} = \tan\alpha \quad \Rightarrow \quad \alpha = \arctan\left(\frac{\tan 30°}{\epsilon}\right) = \arctan\left(\frac{\tan 30°}{0.8}\right) \quad \Rightarrow \quad \boxed{\alpha \approx 35.8°}. \tag{2.3}$$

From this and equation 2.1 we see that

$$v = u\frac{\sin 30°}{\sin\alpha} \quad \Rightarrow \quad \boxed{v \approx 4.27 m/s}. \tag{2.4}$$

## 3 Maximum Momentum

A rocket starts from rest in free space (no gravity). The rocket exhaust speed, $u$, is constant. At what fraction of the initial rocket mass, $m/m_0$ , is the momentum of the rocket a maximum?

**3.0 solution**

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = \dot{m}v + m\dot{v} = \dot{m}v + (-u\dot{m}) = \dot{m}(u - v) \tag{3.1}$$

where we used the rocket equation $m\dot{v} = -u\dot{m}$. We wish to maximize $mv$ in time where $m$ is the mass of the rocket, $v$ is the speed of the rocket, and $v = v'$ is the speed of the rocket that maximizes $mv$, so

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}(mv)\right|_{v=v'} = 0 \quad \Rightarrow \quad \dot{m}(u - v') = 0 \quad \Rightarrow \quad v' = u\,. \tag{3.2}$$

From integrating the rocket equation, $m\dot{v} = -u\dot{m}$ with the initial speed $v_0$ set to zero we have

$$\frac{v}{u} = -\ln\frac{m}{m_0} \quad \Rightarrow \quad \frac{m}{m_0} = \mathrm{e}^{-\frac{v}{u}}\,. \tag{3.3}$$

We plug in $v = v' = u$ giving

$$\boxed{\frac{m}{m_0} = \frac{1}{\mathrm{e}} \approx 0.368}\,. \tag{3.4}$$

Now we wish to show that this is indeed a maximum by showing that

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}(mv)\right|_{v=v'} < 0\,. \tag{3.5}$$

We have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(mv) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{m}v + m\dot{v}) = \ddot{m}v + \dot{m}\dot{v} + \dot{m}\dot{v} + m\ddot{v} = \ddot{m}v + m\ddot{v} + 2\dot{m}\dot{v}\,. \tag{3.6}$$

Differentiating the rocket equation, $m\dot{v} = -u\dot{m}$, with respect to time gives

$$\dot{m}\dot{v} + m\ddot{v} = -\ddot{m}u \tag{3.7}$$

Comparing this with equation 3.6 gives

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(mv) = \ddot{m}v + \dot{m}\dot{v} - \ddot{m}u = \dot{m}\dot{v} + \ddot{m}(v - u)\,. \tag{3.8}$$

So

$$\left.\frac{\mathrm{d}^2}{\mathrm{d}t^2}(mv)\right|_{v=v'=u} = \dot{m}\dot{v} \tag{3.9}$$

which is less than zero, so we have a maximum.

## 4 Energy from a Rocket Engine

A rocket in outer space starts from rest and accelerates with constant acceleration $a$, until its final speed is $v$. The initial mass of the rocket is $m_0$. The relative rocket exhaust speed is the constant $u$. How much work does the rocket's engine do? Include the work on the expended mass and the rocket.

**4.0 solution**

Positive work is done on both the rocket and the expended exhaust mass

$$W = \int F_{\text{rocket}}\, \mathrm{d}x_{\text{rocket}} + \int F_{\text{(expended mass)}}\, \mathrm{d}x_{\text{(expended mass)}}$$

$$= \int F_{\text{rocket}} \frac{\mathrm{d}x_{\text{rocket}}}{\mathrm{d}t}\mathrm{d}t + \int F_{\text{(expended mass)}} \frac{\mathrm{d}x_{\text{(expended mass)}}}{\mathrm{d}t}\mathrm{d}t\,. \tag{4.1}$$

We have

$$F_{\text{rocket}} = \frac{\mathrm{d}}{\mathrm{d}t}(mv) = \dot{m}v + m\dot{v} = \dot{m}v - \dot{m}u \tag{4.2}$$

where we used the rocket equation, $m\dot{v} = -\dot{m}u$, where $m$ is the mass of the rocket. The impulse applied to the exhausted mass $-\mathrm{d}m$ is

$$F_{\text{(expended mass)}}\mathrm{d}t = -\mathrm{d}m\,(v-u) \quad \Rightarrow \quad F_{\text{(expended mass)}} = -\dot{m}\,(v-u)\,. \tag{4.3}$$

With this, the speed of the rocket being $v$, and the speed of the exhaust being $v-u$, along with equation 4.1 we get

$$W = \int (\dot{m}v - \dot{m}u)\, v\, \mathrm{d}t + \int \left[-\dot{m}\,(v-u)\right](v-u)\,\mathrm{d}t = -\int \dot{m}\left(u^2 - uv\right)\mathrm{d}t - u^2 \int \dot{m}\mathrm{d}t + u\int v\dot{m}\mathrm{d}t$$

$$= -u^2 \int \mathrm{d}m + u \int v\mathrm{d}m \tag{4.4}$$

We can integrate the second integral by parts giving

$$W = -u^2 \int \mathrm{d}m + uv'm\big|_{v'=0}^{v} - u\int m\mathrm{d}v = -u^2\int \mathrm{d}m + uvm - u\int m\dot{v}\mathrm{d}t = -u^2\int \mathrm{d}m + uvm + u\int u\dot{m}\mathrm{d}t$$

$$= -u^2\int \mathrm{d}m + umv + u^2\int \mathrm{d}m = umv \tag{4.5}$$

where we have used the rocket equation $m\dot{v} = -u\dot{m}$. We can plug in the $v$ dependence for $m$

$$m = m_0\, \mathrm{e}^{-\frac{v}{u}} \tag{4.6}$$

giving

$$\Rightarrow \quad \boxed{W = m_0 uv\, \mathrm{e}^{-\frac{v}{u}}}\,. \tag{4.7}$$

It's interesting to note that in general (without requiring constant acceleration) this is the power from the engine. We did not use the fact that the acceleration was constant, but $u$ being constant is required.