1 Effect of the Coriolis Force on an Artillery Projectile

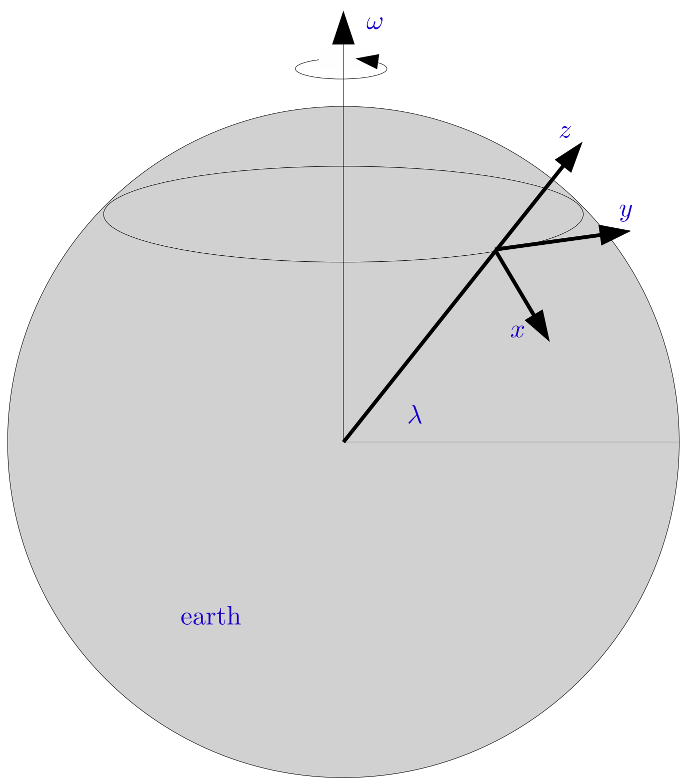
1.1 Lateral Deflection

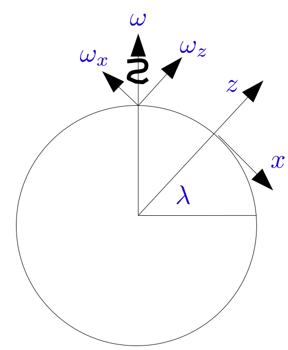
A projectile is fired due east with an angle of inclination to the horizontal of α and initial speed v_0. This takes place on the northern hemisphere at a latitude of λ. Show that the southward (lateral) deflection of the projectile can be approximated as

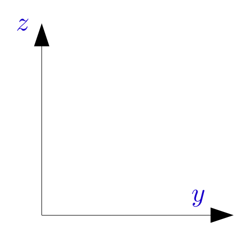
$$d = \frac{4v_0^3}{g^2}\,\omega \sin\lambda\,\sin^2\alpha\,\cos\alpha \tag{1.1}$$

where g is the acceleration due to gravity near the surface of the earth, and ω is the angular frequency of rotation of the earth. Neglect air resistance.

1.1 solution







The equations of motion for a free fall object with mass m and with the Coriolis forces can be expressed as

$$m\ddot{\vec{r}} = -mg\hat{z} - 2m\vec{\omega}\times\dot{\vec{r}} \quad\Rightarrow\quad \ddot{\vec{r}} = -g\hat{z} - 2\vec{\omega}\times\dot{\vec{r}}\,. \tag{1.2}$$

Because we have faster motion in the z-y plane, we can ignore all the components of the Coriolis force except for forces in the x-direction, and we have the usual free fall equations of motion for the y and z directions, giving us

$$\ddot{x} = 2\omega\dot{y}\sin\lambda \tag{1.3}$$
$$\dot{y} = v_0\cos\alpha \tag{1.4}$$
$$\ddot{z} = -g\,. \tag{1.5}$$

Integrating equation 1.3 gives

$$\dot{x}(t) = 2\omega v_0\cos\alpha\,(\sin\lambda)\;t \quad\Rightarrow\quad x(t) = \omega v_0\cos\alpha\,(\sin\lambda)\;t^2 \tag{1.6}$$

where we have used the initial conditions that $\dot{x}(0) = 0$ and $x(0) = 0$. To get the deflection in the x-direction we get the the time of flight, t_f, from the z motion and plug that into $x(t)$ to give us $x(t_f) = d$. So integrating equation 1.5 gives

$$z(t) = v_0 (\sin\alpha)\, t - \frac{1}{2} g t^2 \tag{1.7}$$

where we have used the initial conditions $\dot{z}(0) = v_0 \sin\alpha$ and $z(0) = 0$. We get the the time of flight, t_f, from setting $z(t_f) = 0$ giving $t_f = \frac{2v_0 \sin\alpha}{g}$. So

$$d = x(t_f) = \omega v_0 \cos\alpha\, (\sin\lambda) \left(\frac{2v_0 \sin\alpha}{g}\right)^2 \quad \Rightarrow \quad \boxed{d = \frac{4v_0^3}{g^2}\, \omega \sin\lambda\, \sin^2\alpha\, \cos\alpha}. \tag{1.8}$$

1.2 Is this an Important Effect for Artillery?

A howitzer has a muzzle velocity of about 680 m/s. If it fires a projectile due east with an angle of inclination of 45° and at a latitude of 45°, what will be the range, R, of this projectile and the southward deflection projectile, d?

1.2 solution

The range, R, is $y(t)$ with $t = t_f = \frac{2v_0 \sin\alpha}{g}$. Integrating equation 1.4 we get

$$y(t) = v_0 (\cos\alpha)\, t \tag{1.9}$$

where we used the initial condition $y(0) = 0$. So

$$R = v_0 (\cos\alpha) \left(\frac{2v_0 \sin\alpha}{g}\right) = \frac{2v_0^2 \cos\alpha \sin\alpha}{g} = \frac{2\,(680\,\mathrm{m/s})^2\, 0.5}{9.8\,\mathrm{m/s^2}} \quad \Rightarrow \quad \boxed{R \approx 47{,}180\,\mathrm{m}}. \tag{1.10}$$

From equation 1.8

$$d = \frac{4v_0^3}{g^2}\, \omega \sin\lambda\, \sin^2\alpha\, \cos\alpha = \frac{4\,(680\,\mathrm{m/s})^3}{(9.8\,\mathrm{m/s^2})^2}\, \frac{2\,\pi}{24 \times 3600\, s}\, \frac{1}{\sqrt{2}}\, \frac{1}{2}\, \frac{1}{\sqrt{2}} \quad \Rightarrow \quad \boxed{d \approx 238\,\mathrm{m}}. \tag{1.11}$$

Looks like it may be important.

2 Effect of the Coriolis Force on a Projectile Going Straight Up and Down

A projectile is fired straight up and reaches a maximum height of h. This takes place on the northern hemisphere at a latitude of λ. **(a)** Show that the projectile lands distance of approximately

$$d = \frac{8}{3} \sqrt{\frac{2h^3}{g}}\, \omega\, \cos\lambda \tag{2.1}$$

from where is was launched, where g is the acceleration due to gravity, and ω is the angular frequency of rotation of the earth. Neglect air resistance. **(b)** What is the direction that the projectile is deflected.

2.0 solution

The equations of motion for a free fall object with mass m and with the Coriolis forces can be expressed as

$$m\ddot{\vec{r}} = -mg\hat{z} - 2m\vec{\omega}\times\dot{\vec{r}} \quad \Rightarrow \quad \ddot{\vec{r}} = -g\hat{z} - 2\vec{\omega}\times\dot{\vec{r}}\,. \tag{2.2}$$

Because we have faster motion in the z-y plane, we can ignore all the components of the Coriolis force in the z-direction, and so we have the usual free fall equations of motion for the z directions, the considerable Coriolis force in the y-direction comes from a component of velocity in the z-direction, and there are no Coriolis force in the x-direction coming from a component of velocity in the z-direction, so we can ignore the x-direction motion. So we will use the equations of motion

$$\ddot{y} = -2\omega\,\dot{z}\,\cos\lambda \tag{2.3}$$

$$\ddot{z} = -g\,. \tag{2.4}$$

Integrating equation 2.4 we get

$$\dot{z}(t) = v_0 - gt \tag{2.5}$$

$$z(t) = v_0 t - \frac{1}{2}gt^2 \tag{2.6}$$

where we have used the initial conditions $\dot{z}(0) = v_0$ and $z(0) = 0$. Using $\dot{z}$ from equation 2.5 we integrate equation 2.3

$$\dot{y} = -2\omega\,\cos\lambda \int (v_0 - gt)\,\mathrm{d}t = -2\omega\,\cos\lambda\left(v_0 t - \frac{1}{2}gt^2\right) \tag{2.7}$$

where we have used the initial condition $\dot{y}(0) = 0$. Integrating again we get

$$y = -2\omega\,\cos\lambda \int \left(v_0 t - \frac{1}{2}gt^2\right)\mathrm{d}t = -2\omega\,\cos\lambda\left(\frac{1}{2}v_0 t^2 - \frac{1}{6}gt^3\right) \tag{2.8}$$

where we have used the initial condition $y(0) = 0$. The y-deflection of the projectile when it lands can be gotten by plugging in the time of flight into $y(t)$. We get the time of flight from solving $z(t = t_f) = 0$ from equation 2.6 giving

$$t_f = \frac{2v_0}{g}\,. \tag{2.9}$$

So the y-deflection of the projectile when is lands is

$$d = y(t_f) = -2\omega\,\cos\lambda\left(\frac{1}{2}v_0 t_f^2 - \frac{1}{6}gt_f^3\right) = -2\omega\,\cos\lambda\left[\frac{1}{2}v_0\left(\frac{2v_0}{g}\right)^2 - \frac{1}{6}\,g\left(\frac{2v_0}{g}\right)^3\right]$$

$$= -2\omega\,\cos\lambda\left(2 - \frac{4}{3}\right)\frac{v_0^3}{g^2} = -\frac{4}{3}\omega\,\cos\lambda\,\frac{v_0^3}{g^2}\,. \tag{2.10}$$

We see that the projectile is deflected in the $-y$ direction, which is west. We can put this in terms of the height, h, that the height the projectile goes, which is the z-position when $\dot{z}(t = t_t) = 0$. So from equation 2.5

$$t_t = \frac{v_0}{g}\,. \tag{2.11}$$

So

$$h = z(t_t) = v_0\frac{v_0}{g} - \frac{1}{2}g\left(\frac{v_0}{g}\right)^2 = \frac{1}{2}\frac{v_0^2}{g} \quad \Rightarrow \quad v_0 = \sqrt{2\,g\,h}\,. \tag{2.12}$$

So from this and equation 2.10

$$\boxed{(\mathbf{a})\ d = \frac{8}{3}\sqrt{\frac{2h^3}{g}}\,\omega\,\cos\lambda \qquad (\mathbf{b})\ \text{West } (-\hat{y})}\,. \tag{2.13}$$