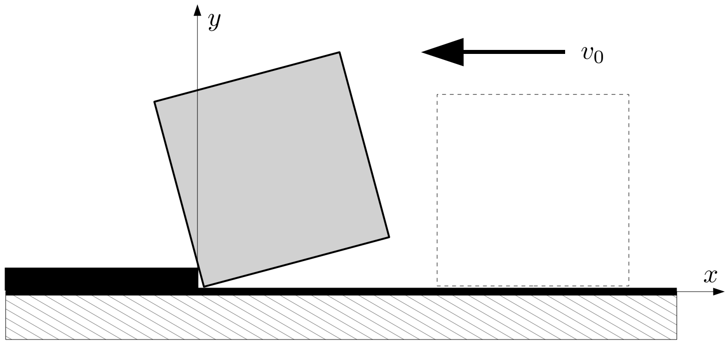
1 A Sliding Block Hits a Small Step

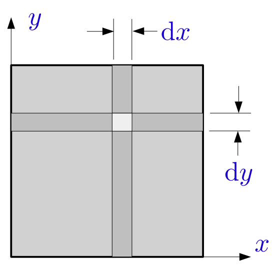


A uniform cube with side length a and mass M slides on a flat frictionless plane. The cube initially has a speed of v_0 and is initially not spinning. The lower leading edge of the cube aligns with a small step in the plane (along the z-direction). The lower leading edge of the cube upon hitting the step comes to an abrupt stop as the cube rotates about this edge (at the step), as shown in the figure above.

1.1 Moment of Inertia

Find the moment of inertia of the block about an edge of the cube, I_e.

1.1 solution



Let the z axis be the direction of rotation, and the cube is in the positive x and y quadrant. The moment of inertia of a piece of the cube that is a stick with square cross-section dx by dy of length a, at position x , y is

$$\mathrm{d}I_e = \left(x^2+y^2\right)\mathrm{d}m = \left(x^2+y^2\right)\rho\, a\,\mathrm{d}x\,\mathrm{d}y = \left(x^2+y^2\right)\left(\frac{M}{a^3}\right) a\,\mathrm{d}x\,\mathrm{d}y \tag{1.1}$$

where we introduced ρ as the mass density of the cube. We can integrate this to give

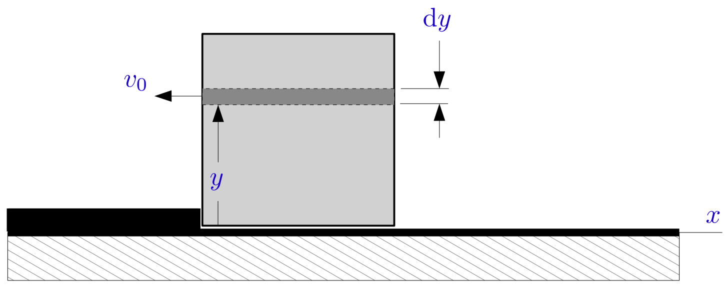
$$I_e = \int_{y=0}^{a}\int_{x=0}^{a}\left(x^2+y^2\right)\left(\frac{M}{a^3}\right) a\,\mathrm{d}x\,\mathrm{d}y = \frac{M}{a^2}\int_{y=0}^{a}\int_{x=0}^{a}\left(x^2+y^2\right)\mathrm{d}x\,\mathrm{d}y = \frac{M}{a^2}\int_{y=0}^{a}\left(\frac{a^3}{3}+ay^2\right)\mathrm{d}y$$

$$= \frac{M}{a^2}\left(\frac{a^3}{3}a + a\frac{a^3}{3}\right) \quad \Rightarrow \quad \boxed{I_e = \frac{2}{3}Ma^2} \tag{1.2}$$

1.2 Angular Speed

Find the angular speed of the cube just after it hits the step, ω_1. Answer in terms of v_0 and a. Hint: Use conservation of angular momentum.

1.2 solution



The angular momentum about the step (and block edge) , L_z, is conserved. Let y be along the vertical direction with $y = 0$ being at the step. L_z before the collision, $L_{z\,0}$, is the sum of square sheets of length dy,

$$L_{z\,0} = \int_{y=0}^{a} \mathrm{d}m\, v_0\, y = \int_{y=0}^{a} \lambda \mathrm{d}y\, v_0\, y = \frac{M}{a} v_0 \int_{y=0}^{a} y\, \mathrm{d}y = \frac{M}{a} v_0 \left(\frac{a^2}{2}\right) = \frac{1}{2} M v_0\, a \tag{1.3}$$

where we have introduced the linear mass density of the cube along the y direction, $\lambda = M/a$. There is no torque on the cube about the edge at the step, so we can equate L_z before the collision, $L_{z\,0}$, to L_z after the collision when the cube is just starting to rotate about the step giving

$$\frac{1}{2} M\, v_0\, a = I_e\, \omega_1 = \frac{2}{3} M a^2 \omega_1 \quad \Rightarrow \quad \boxed{\omega_1 = \frac{3\, v_0}{4\, a}}. \tag{1.4}$$

$L_{z\,0}$ can also be calculated from the sum of the angular momentum about the center-of-mass (which is zero) plus the angular momentum treating the cube as a point mass which is

$$\vec{L} = M \vec{R} \times \vec{v}_0 = M \frac{a}{2} v_0 \hat{z} \tag{1.5}$$

where $\vec{R}$ is the position of the center-of-mass of the cube. This gives the same result as in equation 1.3.

1.3 Minimum Speed

Find the minimum initial speed, $v_{0\,\mathrm{min}}$, that the cube needs for the cube to roll over the small step. Answer in terms of a and g (the acceleration due to gravity).

1.3 solution

After the collision, as the cube rotates about the step, energy will be conserved. Equating the energy just after the collision and to the energy when the center-of-mass of the cube is above the step and not moving, and using equations 1.2 and 1.4 gives the minimum initial speed like so

$$\frac{1}{2} I_e \omega_{1\,\mathrm{min}}^2 = M g\, \Delta h \quad \Rightarrow \quad \frac{1}{2}\left(\frac{2}{3} M a^2\right)\left(\frac{3\, v_{0\,\mathrm{min}}}{4\, a}\right)^2 = M g \left(\frac{\sqrt{2} a}{2} - \frac{a}{2}\right) \tag{1.6}$$

where $\Delta h = \frac{\sqrt{2}a}{2} - \frac{a}{2}$ is the change in the height of the center-of-mass of the cube.

$$\Rightarrow \quad \boxed{v_{0\ \mathrm{min}} = \sqrt{\frac{8\left(\sqrt{2}-1\right)}{3}\, g\, a}}. \tag{1.7}$$