1 Inertia Tensor

A rigid body is made from 8 particles all with mass m. There is one particle at each corner of a cube of side length a. Massless sticks are used to hold the particles together as a rigid body.

1.1 About a Corner of the Cube

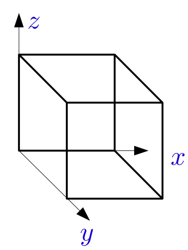
Show that the inertia tensor for rotations about a corner of the cube, with coordinate axes along edges of the cube, is given by

$$\mathbf{I} = ma^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix} . \tag{1.1}$$

1.1 solution

We'll use the form

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} . \tag{1.2}$$



We look at all eight particles at the vertices of a cube of side a, many of which do not contribute to terms. It's just record keeping. Six particles contribute to I_{xx},

$$I_{xx} = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = ma^2 \left(1+2+1+1+2+1\right) = 8ma^2 . \tag{1.3}$$

From the symmetry that interchanging x, y, and z in any way will not change the geometry we have

$$I_{yy} = I_{zz} = I_{xx} = 8ma^2 . \tag{1.4}$$

Only two particles contribute to I_{xy},

$$I_{xy} = -\sum_\alpha m_\alpha \left(x_\alpha y_\alpha\right) = -ma^2 \left(1+1\right) = -2ma^2 . \tag{1.5}$$

From the symmetry that interchanging x, y, and z in any way will not change the geometry we have

$$I_{zy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{xy} = -2ma^2 . \tag{1.6}$$

So the result is

$$\boxed{\mathbf{I} = ma^2 \begin{bmatrix} 8 & -2 & -2 \\ -2 & 8 & -2 \\ -2 & -2 & 8 \end{bmatrix}} . \tag{1.7}$$

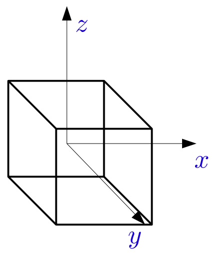
1.2 About the Center of the Cube

Find the inertia tensor for rotations about the center of the cube with coordinate axes parallel to edges of the cube.

1.2 solution

We'll use the form

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} . \tag{1.8}$$



We look at all eight particles at the vertices of a cube of side a, many of which do not contribute to terms. It's just record keeping. Eight particles contribute to I_{xx}, all with $y_\alpha^2 + z_\alpha^2 = a^2/2$, giving

$$I_{xx} = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = 8\left[m\left(\frac{a^2}{2}\right)\right] = 4ma^2 . \tag{1.9}$$

From the symmetry that interchanging x, y, and z in any way will not change the geometry we have

$$I_{yy} = I_{zz} = I_{xx} = 4ma^2 . \tag{1.10}$$

Eight particles contribute to I_{xy},

$$I_{xy} = -\sum_\alpha m_\alpha \left(x_\alpha y_\alpha\right) = -ma^2\left(\frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} - \frac{1}{4}\right) = 0 . \tag{1.11}$$

From the symmetry that interchanging x, y, and z in any way will not change the geometry we have

$$I_{zy} = I_{xz} = I_{yx} = I_{yz} = I_{zx} = I_{xy} = 0 . \tag{1.12}$$

One can also argue that from the reflection symmetry of the body about the x axis, that all single x terms (I_{xy}, I_{yx}, I_{xz}, and I_{zx}) are zero. The same thing can be said for the y and z axis. So the result is

$$\boxed{\mathbf{I} = ma^2 \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}} . \tag{1.13}$$

2 Inertia Tensor for a Solid Cone

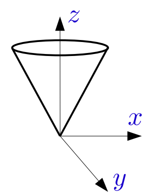
A uniform solid cone has a mass M, a base radius R, and a height of h. The z axis is along the axis of symmetry of the cone. The tip of the cone is at the point of rotation which is at the origin. Show that the moment of inertia for this cone can be written as

$$\mathbf{I} = \frac{3}{20}M \begin{bmatrix} R^2 + 4h^2 & 0 & 0 \\ 0 & R^2 + 4h^2 & 0 \\ 0 & 0 & 2R^2 \end{bmatrix} . \tag{2.1}$$

2.0 solution

We'll use the form

$$\mathbf{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} . \tag{2.2}$$



The equation of the outer surface of the cone is

$$\left(\frac{Rz}{h}\right)^2 = x^2 + y^2 \tag{2.3}$$

where z is limited to be between 0 and h.

$$I_{xx} = \sum_\alpha m_\alpha \left(y_\alpha^2 + z_\alpha^2\right) = \int\int\int \left(y^2+z^2\right)\rho\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{z=0}^{h}\int_{y=-\frac{Rz}{h}}^{\frac{Rz}{h}}\int_{x=-\sqrt{\left(\frac{Rz}{h}\right)^2-y^2}}^{\sqrt{\left(\frac{Rz}{h}\right)^2-y^2}} \left(y^2+z^2\right)\left(\frac{3M}{\pi R^2 h}\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$= \frac{12M}{\pi R^2 h}\int_{z=0}^{h}\int_{y=0}^{\frac{Rz}{h}}\int_{x=0}^{\sqrt{\left(\frac{Rz}{h}\right)^2-y^2}} \left(y^2+z^2\right)\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{12M}{\pi R^2 h}\int_{z=0}^{h}\int_{y=0}^{\frac{Rz}{h}}\left(\sqrt{\left(\frac{Rz}{h}\right)^2 - y^2}\right)\left(y^2+z^2\right)\mathrm{d}y\,\mathrm{d}z\,. \tag{2.4}$$

Looking up the y integral we get

$$I_{xx} = \frac{12M}{\pi R^2 h}\int_{z=0}^{h}\left\{\left[\frac{y}{4}\sqrt{\left[\left(\frac{Rz}{h}\right)^2 - y^2\right]^3} + \frac{\left(\frac{Rz}{h}\right)^2}{8}\left(y\sqrt{\left(\frac{Rz}{h}\right)^2 - y^2} + \left(\frac{Rz}{h}\right)^2 \sin^{-1}\frac{y}{\frac{Rz}{h}}\right)\right]\Bigg|_{y=0}^{\frac{Rz}{h}}\right\}\mathrm{d}z + \tag{2.5}$$

$$\frac{12M}{\pi R^2 h}\int_{z=0}^{h}\left[\frac{1}{2}\left(y\sqrt{\left(\frac{Rz}{h}\right)^2 - y^2} + \left(\frac{Rz}{h}\right)^2\sin^{-1}\frac{y}{\frac{Rz}{h}}\right)\Bigg|_{y=0}^{\frac{Rz}{h}}\right] z^2\mathrm{d}z$$

$$= \frac{12M}{\pi R^2 h}\int_{z=0}^{h}\left[\frac{\left(\frac{Rz}{h}\right)^4}{8}\left(\frac{\pi}{2}\right) + \frac{\left(\frac{Rz}{h}\right)^2}{2}\left(\frac{\pi}{2}\right)z^2\right]\mathrm{d}z = 3M\left(\frac{R^2}{4h^5} + \frac{1}{h^3}\right)\int_{z=0}^{h} z^4\mathrm{d}z$$

$$\Rightarrow \quad I_{xx} = \frac{3}{20}M\left(R^2 + 4h^2\right) . \tag{2.6}$$

From symmetry

$$I_{xx} = I_{yy} = \frac{3}{20}M\left(R^2+4h^2\right) . \tag{2.7}$$

We can get I_{zz} by summing over disks with moment of inertia $\frac{1}{2}m_\alpha r_\alpha^2$ for each disk of height $\mathrm{d}z$, $m_\alpha = \rho\pi r^2\mathrm{d}z$, $r_\alpha^2 = R^2z^2/h^2$, and $\rho = \frac{3M}{\pi R^2 h}$ giving

$$I_{zz} = \int_{z=0}^{h}\frac{1}{2}\left(\frac{3M}{\pi R^2 h}\right)\pi\frac{R^2z^2}{h^2}\,\mathrm{d}z\,\frac{R^2z^2}{h^2} = \frac{3MR^2}{2h^5}\int_{z=0}^{h} z^4\,\mathrm{d}z = \frac{3MR^2}{2h^5}\left(\frac{h^5}{5}\right) = \frac{3}{10}MR^2\,.$$

The reflection symmetry about the x axis make I_{xy}, I_{yx}, I_{xz}, and I_{zx} all zero. The reflection symmetry about the y axis make I_{yx}, I_{xy}, I_{yz}, and I_{zy} all zero. So all off diagonal elements zero,

$$I_{xy} = I_{yx} = I_{xz} = I_{zx} = I_{yz} = I_{zy} = 0\,. \tag{2.8}$$

So the final result is

$$\boxed{\mathbf{I} = \frac{3}{20} M \begin{bmatrix} R^2 + 4h^2 & 0 & 0 \\ 0 & R^2 + 4h^2 & 0 \\ 0 & 0 & 2R^2 \end{bmatrix}}\,. \tag{2.9}$$