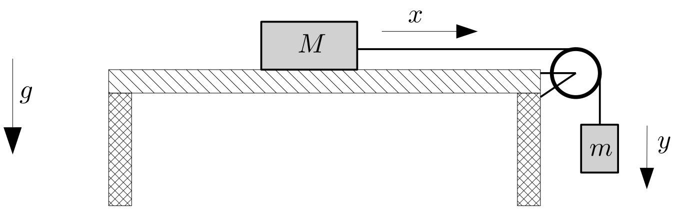
This does not cover all topics that could be on the final exam. Topics that are covered in the homework and quizzes are possible topics that may be on the final exam. Recent course topics are not covered in this homework.

1 Blocks with Pulley

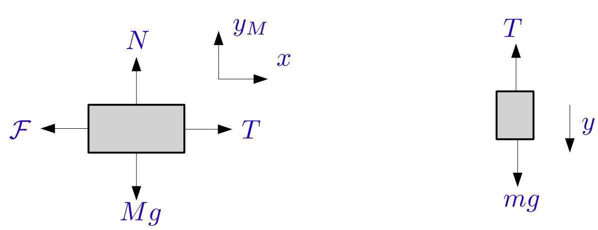


A block of mass M slides on the top of a table. The coefficient of static friction between the block and the table is μ_s. The coefficient of kinetic friction between the block and the table is μ_k. The pulley is massless and frictionless. A massless stretch-less string connects the sliding block to a hanging weight. The displacement of the block is given by x. The displacement of the hanging weight is given by y.

1.1 Free Body Diagrams

Draw a free body diagram for the block and the hanging weight.

1.1 solution



1.2 Minimum Weight Mass

What is the minimum (limiting) hanging weight mass, $m_{\min}$, that will cause the sliding block to start to move. Answer in terms of M, and μ_s.

1.2 solution

Applying Newton's second law to the sliding block and the hanging weight gives

$$\sum_M F_{y_M} = N - Mg = 0 \quad \Rightarrow \quad N = Mg \tag{1.1}$$

$$\sum_M F_x = T - \mathcal{F} = M\ddot{x} \tag{1.2}$$

$$\sum_m F_y = mg - T = m\ddot{y}\,. \tag{1.3}$$

Since the string is massless and does not stretch we have the constraint equation

$$\ddot{x} = \ddot{y}\,. \tag{1.4}$$

Solving for $\ddot{x}$ from all of the above equations, while eliminating $\ddot{y}$ and T gives

$$(M+m)\,\ddot{x} = mg - \mathcal{F}\,. \tag{1.5}$$

Since the block is not moving yet we use

$$\mathcal{F} \le \mu_s N\,. \tag{1.6}$$

$\mathcal{F}$ will keep the block at rest until it reaches it's maximum value of $\mu_s N = \mu_s Mg$. So the when the block just starts to move m will be the minimum needed mass, $m_{\rm min}$.

$$m_{\rm min} g - \mathcal{F}_{\rm max} = 0 \quad \Rightarrow \quad m_{\rm min} g - \mu_s Mg = 0 \quad \Rightarrow \quad \boxed{m_{\rm min} = \mu_s M}\,. \tag{1.7}$$

1.3 Acceleration

When the block is moving, what is acceleration of the sliding block, $\ddot{x}$, as a function of M, m, g, and μ_k.

1.3 solution

When the block is moving to the right

$$\mathcal{F} = \mu_k N = \mu_k Mg\,. \tag{1.8}$$

With this and equation 1.5 we get

$$(M+m)\,\ddot{x} = mg - \mu_k Mg \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{m - \mu_k M}{M+m}\, g}\,. \tag{1.9}$$

This assumes that the block is sliding to the right. When the block is moving to the left

$$\mathcal{F} = -\mu_k N \quad \Rightarrow \quad \boxed{\ddot{x} = \frac{m + \mu_k M}{M+m}\, g}\,. \tag{1.10}$$

2 Power Into a Simple Harmonic Oscillator

A driven damped simple harmonic oscillator consists of a mass m, connected to a spring with spring constant k, and linear damping constant b ($F_{\rm damping} = -b\dot{x}$). A driving force acts on the mass with a force of $F(t) = F_0 \cos\omega t$ along the direction of the oscillation of mass.

2.1 Average Power In

In terms of the given parameters, find the average power $\langle P\rangle$ ($P = \vec{F}\cdot\vec{v}$) that the driving force applies to the oscillator over one cycle of the oscillator, when the oscillator is in steady state motion.

2.1 solution

The equation of motion for this driven oscillator can be written as

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos\omega t \quad \Rightarrow \quad \ddot{x} + 2\frac{b}{2m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t\,. \tag{2.1}$$

From many texts we see that the steady state solution to this can be written as

$$x(t) = D\cos(\omega t - \delta) \tag{2.2}$$

$$D \equiv \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} \tag{2.3}$$

and

$$\delta \equiv \tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right), \tag{2.4}$$

where $\omega_0^2 \equiv \frac{k}{m}$, $\beta \equiv \frac{b}{2m}$. The power supplied by the driver to the oscillator at steady state is

$$P = (F_0\cos\omega t)\,\dot{x}(t) = F_0\cos\omega t\,[-\omega D\sin(\omega t - \delta)] \quad \Rightarrow \quad \langle P\rangle = -\omega D F_0 \frac{\omega}{2\pi}\int_{t=0}^{\frac{\omega}{2\pi}} \cos\omega t\,\sin(\omega t - \delta)\,\mathrm{d}t$$

$$= -\omega D F_0 \frac{1}{2\pi}\int_{t=0}^{\frac{\omega}{2\pi}} \cos\omega t\,(\sin\omega t\,\cos\delta - \cos\omega t\,\sin\delta)\,\mathrm{d}(\omega t)$$

$$= -\omega D F_0\left(0 - \frac{1}{2}\sin\delta\right) \quad \Rightarrow \quad \langle P\rangle = \frac{1}{2}\omega D F_0\sin\delta = \frac{1}{2}\omega D F_0 \sin\left[\tan^{-1}\left(\frac{2\omega\beta}{\omega_0^2 - \omega^2}\right)\right]$$

$$= \frac{1}{2}\omega D F_0 \frac{2\omega\beta}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}} = \beta\omega^2 F_0 \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}\,\frac{1}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}}$$

$$= \frac{\frac{b}{2m}\omega^2\frac{F_0^2}{m}}{\left(\omega_0^2 - \omega^2\right)^2 + 4\omega^2\beta^2}$$

$$\Rightarrow \quad \boxed{\langle P\rangle = \frac{\frac{b\omega^2 F_0^2}{2m^2}}{\left(\frac{k}{m} - \omega^2\right)^2 + 4\omega^2\beta^2}}\,. \tag{2.5}$$

2.2 Power Resonance

Find, ω_r, the value of the driving angular frequency, ω, that maximizes this average power, $\langle P\rangle$.

2.2 solution

For $\langle P\rangle$ to be a maximum

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\langle P\rangle\bigg|_{\omega=\omega_r} = 0 \quad \Rightarrow \quad \frac{\frac{b\omega_r F_0^2}{m^2}\left[\left(\frac{k}{m} - \omega_r^2\right)^2 + 4\omega_r^2\beta^2\right] - \frac{b\omega_r^2 F_0^2}{2m^2}\left[2\left(\frac{k}{m} - \omega_r^2\right)(-2\omega_r) + 8\omega_r\beta^2\right]}{\left[\left(\frac{k}{m} - \omega_r^2\right)^2 + 4\omega_r^2\beta^2\right]^2} = 0$$

$$\Rightarrow \quad \omega_r\left[\left(\frac{k}{m} - \omega_r^2\right)^2 + 2\omega_r^2\left(\frac{k}{m} - \omega_r^2\right)\right] = 0\,. \tag{2.6}$$

The root of interest is

$$\boxed{\omega_r = \sqrt{\frac{k}{m}} = \omega_0}\,, \tag{2.7}$$

which is the angular frequency of the undriven, undamped oscillator.

Another condition that $\langle P\rangle$ to be a maximum, at $\omega = \omega_r$, is that

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\langle P\rangle\bigg|_{\omega=\omega_r} < 0\,. \tag{2.8}$$

We have

$$\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\langle P\rangle &= \frac{bF_0^2}{m^2}\frac{\mathrm{d}}{\mathrm{d}\omega}\left\{\frac{\omega\left(\frac{k}{m}-\omega^2\right)^2 + 2\omega^3\left(\frac{k}{m}-\omega^2\right)}{\left[\left(\frac{k}{m}-\omega^2\right)^2+4\omega^2\beta^2\right]^2}\right\} \\
&= \frac{bF_0^2}{m^2}\left\{\frac{\left[\left(\frac{k}{m}-\omega^2\right)^2 + \omega 2\left(\frac{k}{m}-\omega^2\right)(-2\omega) + 6\omega^2\left(\frac{k}{m}-\omega^2\right) + 2\omega^3(-2\omega)\right]\left[\left(\frac{k}{m}-\omega^2\right)^2+4\omega^2\beta^2\right]^2}{\left[\left(\frac{k}{m}-\omega^2\right)^2+4\omega^2\beta^2\right]^4}\right. \\
&\qquad \left. -\frac{\left[\omega\left(\frac{k}{m}-\omega^2\right)^2 + 2\omega^3\left(\frac{k}{m}-\omega^2\right)\right]2\left[\left(\frac{k}{m}-\omega^2\right)^2+4\omega^2\beta^2\right]\left[2\left(\frac{k}{m}-\omega^2\right)(-2\omega)+8\omega\beta^2\right]}{\left[\left(\frac{k}{m}-\omega^2\right)^2+4\omega^2\beta^2\right]^4}\right\}.
\end{aligned} \tag{2.9}$$

Setting $\omega = \omega_r = \sqrt{\frac{k}{m}}$ gives

$$\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\langle P\rangle\bigg|_{\omega=\omega_r} = \frac{bF_0^2}{m^2}\left(\frac{-64\omega_r^{16}\beta^4}{(4\omega_r^2\beta^2)^4}\right) = -\frac{bF_0^2}{m^2}\left(16\frac{k^4}{m^4\beta^4}\right) \tag{2.10}$$

which is less than zero, so $\omega = \frac{k}{m}$ maximumizes $\langle P\rangle$, the average power over a cycle.

3 Find a Force from an Orbit

Find the central force, $F(r)$, that allows a particle to move in a spiral orbit given by $r = k\theta$, where k is a constant.

3.0 solution

From Thornton and Marion (equation 8.21), the force, $F(r)$, can be found from the ordinary differential equation

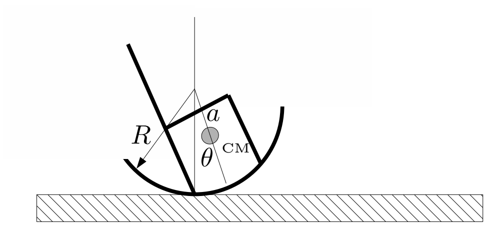
$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r)\,, \tag{3.1}$$

where l is the constant angular momentum, and $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass of the two particles. So the force, $F(r)$, is

$$\begin{aligned}
F(r) &= -\frac{l^2}{k\mu r^2}\left[\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{\theta}\right)+\frac{1}{\theta}\right] = -\frac{l^2}{k\mu r^2}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-\frac{1}{\theta^2}\right)+\frac{1}{\theta}\right] = -\frac{l^2}{k\mu r^2}\left(2\frac{1}{\theta^3}+\frac{1}{\theta}\right) \\
&= -\frac{l^2}{k\mu r^2}\left(2\frac{k^3}{r^3}+\frac{k}{r}\right) \quad\Rightarrow\quad F(r) = -\frac{l^2}{k\mu r^2}\left(2\frac{k^3}{r^3}+\frac{k}{r}\right) \quad\Rightarrow\quad \boxed{F(r) \propto -2\frac{k^2}{r^5}-\frac{1}{r^3}}
\end{aligned} \tag{3.2}$$

and we need more information (like l and μ) to give the force that will act in a particular case.

4 Rocking Chair

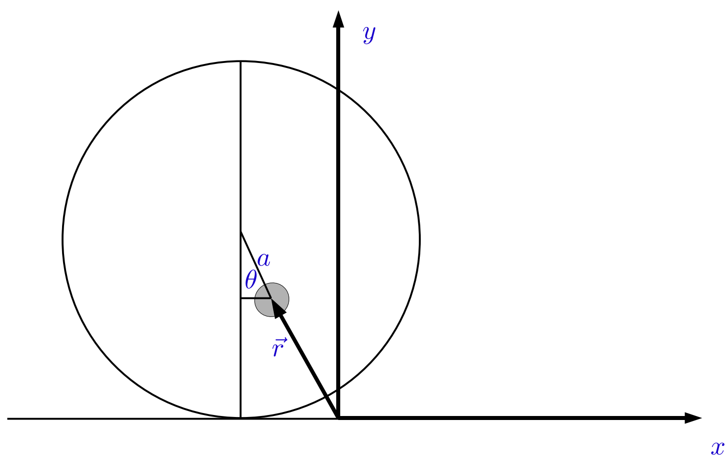


A rocking chair rolls (rocks) without slipping. The radius of the rocker is R. The total mass of the rocking chair is M. The distance from the center of the rocker circle to the center-of-mass (CM) is a. The momentum-of-inertia of the rocking chair about the center-of-mass is I. No one is sitting in the rocking chair.

4.1 Lagrangian

Find the Lagrangian $L(\theta, \dot{\theta})$, where θ is the angle of rotation of the rocking chair measured from the equilibrium position.

4.1 solution



To calculate the kinetic energy T we will need $(\dot{\vec{r}})^2$ which we get from $\vec{r}$ by looking at the above figure

$$\vec{r} = (-R\theta + a\sin\theta)\,\hat{x} + (R - a\cos\theta)\,\hat{y} \quad \Rightarrow \quad \dot{\vec{r}} = \left(-R\dot{\theta} + a\dot{\theta}\cos\theta\right)\hat{x} + \left(a\dot{\theta}\sin\theta\right)\hat{y}$$

$$\Rightarrow \quad \left(\dot{\vec{r}}\right)^2 = R^2\dot{\theta}^2 - 2aR\dot{\theta}^2\cos\theta + a^2\dot{\theta}^2\,. \tag{4.1}$$

So the Lagrangian L is

$$L = T - U = T_{\text{CM}} + T_{\text{rot}} - U \tag{4.2}$$

where we have written the kinetic energy T, in terms of the kinetic energy due to the translation of the center-of-mass, T_{CM}, and the kinetic energy due to the rotation about the center-of-mass, T_{rot}. Continuing,

$$L = T_{\text{CM}} + T_{\text{rot}} - U = \frac{1}{2}M\left(R^2\dot{\theta}^2 - 2aR\dot{\theta}^2\cos\theta + a^2\dot{\theta}^2\right) + \frac{1}{2}I\dot{\theta}^2 - (-Mga\cos\theta)$$

$$\Rightarrow \quad \boxed{L(\theta, \dot{\theta}) = \frac{1}{2}MR^2\dot{\theta}^2 + \frac{1}{2}Ma^2\dot{\theta}^2 - MaR\dot{\theta}^2\cos\theta + \frac{1}{2}I\dot{\theta}^2 + Mga\cos\theta}\,. \tag{4.3}$$

4.2 Differential Equation of Motion for θ

Find the differential equation of motion for θ. You may answer in terms of $\ddot{\theta}$, $\dot{\theta}$, θ, M, a, R, I, and g (the acceleration due to gravity).

4.2 solution

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad MaR\dot{\theta}^2 \sin\theta - Mga\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(MR^2\dot{\theta} + Ma^2\dot{\theta} + I\dot{\theta} - 2MaR\dot{\theta}\cos\theta\right)$$

$$\Rightarrow \quad MaR\dot{\theta}^2 \sin\theta - Mga\sin\theta - \left(MR^2\ddot{\theta} + Ma^2\ddot{\theta} + I\ddot{\theta} - 2MaR\ddot{\theta}\cos\theta + 2MaR\dot{\theta}^2\sin\theta\right)$$

$$\Rightarrow \quad \boxed{\left(MR^2 + Ma^2 + I\right)\ddot{\theta} - 2MaR\ddot{\theta}\cos\theta = -MaR\dot{\theta}^2\sin\theta - Mga\sin\theta}\,. \tag{4.4}$$

4.3 Angular Frequency for Small Oscillations

Find the angular frequency, ω_0, for small oscillations about $\theta = 0$. You may answer in terms of M, a, R, I, and g (the acceleration due to gravity).

4.3 solution

For small θ we approximate

$$\sin\theta \approx \theta \qquad \text{and} \qquad \cos\theta \approx 1\,. \tag{4.5}$$

Using this with the equation of motion, 4.4, we get

$$\left(MR^2 + Ma^2 + I\right)\ddot{\theta} - 2MaR\ddot{\theta} = -MaR\dot{\theta}^2\theta - Mga\theta\,. \tag{4.6}$$

$\dot{\theta}^2\theta$ will be of size of order θ^3 and may be ignored. So by using that and simplifying equation 4.6 we have

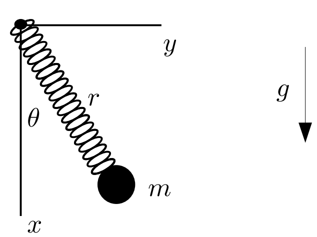
$$\left[M\left(R-a\right)^2 + I\right]\ddot{\theta} = -Mga\theta \quad \Rightarrow \quad \ddot{\theta} = -\frac{Mga}{M\left(R-a\right)^2 + I}\,\theta \tag{4.7}$$

which is the equation of motion for simple harmonic motion with the angular frequency

$$\boxed{\omega_0 = \sqrt{\frac{Mga}{M\left(R-a\right)^2 + I}}}\,. \tag{4.8}$$

It may seem a little odd that for the limiting case when $R = a$ and $I = 0$ that $\omega_0 = \infty$, but this makes some sense when you consider that the velocity of the center of mass would be constrained to be zero at the equilibrium position $\theta = 0$ if $R = a$, from the geometry. So it's not linearizable (simple harmonic) if $R = a$.

5 Springy Pendulum



The springy pendulum shown above has a rest spring length a, a spring constant k, and a bob mass of m. r is the distance from the origin (pivot point) to the bob. Both r and θ can change in time.

5.1 Hamiltonian

Find the generalized momentums p_r and p_θ in terms of the generalized coordinates and velocities, and find the Hamiltonian $H(p_r, r, p_\theta, \theta)$ for this system.

5.1 solution

The Lagrangian is

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}k(r-a)^2 + mgr\cos\theta \tag{5.1}$$

The generalized momentum p_r is

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \Rightarrow \quad \boxed{p_r = m\dot{r}}. \tag{5.2}$$

The generalized momentum p_θ is

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad \Rightarrow \quad \boxed{p_r = mr^2\dot{\theta}}. \tag{5.3}$$

With the above, the Hamiltonian is

$$H = p_r\dot{r} + p_\theta\dot{\theta} - L = 2T - T + U = T + U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}k(r-a)^2 - mgr\cos\theta$$

$$\Rightarrow \quad \boxed{H = \frac{\dot{p_r}^2}{2m} + \frac{\dot{p_\theta}^2}{2mr^2} + \frac{1}{2}k(r-a)^2 - mgr\cos\theta}. \tag{5.4}$$

In this case we see that the Hamiltonian is equal to the total energy.

5.2 Differential Equations of Motion

Find the differential equations of motion for p_r, p_θ, r, and θ, in terms of $\dot{p}_r$, $\dot{p}_\theta$, $\dot{r}$, $\dot{\theta}$, p_r, p_θ, r, θ, m, k, a, and g.

5.2 solution

$$\dot{p_r} = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - kr + ka + mg\cos\theta\,, \tag{5.5}$$

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -mgr\sin\theta\,, \tag{5.6}$$

$$\dot{r} = \frac{\partial H}{\partial p_r} = \frac{p_r}{m}\,, \tag{5.7}$$

and

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2}\,. \tag{5.8}$$

So in summary

$$\boxed{\begin{aligned} \dot{p_r} &= \tfrac{p_\theta^2}{mr^3} - kr + ka + mg\cos\theta \\ \dot{p_\theta} &= -mgr\sin\theta \\ \dot{r} &= \tfrac{p_r}{m} \\ \dot{\theta} &= \tfrac{p_\theta}{mr^2} \end{aligned}} \,. \tag{5.9}$$

6 Inverse Rocket

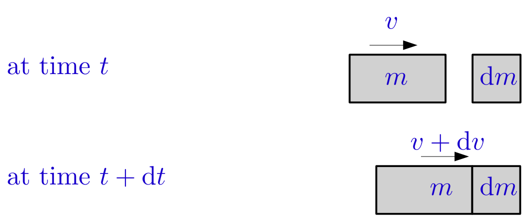
A large abandoned space ship travels through space which is filled with uniformly distributed "space dust", with mass density ρ. The only forces on the space ship are from the dust that collects on the ship as it goes through the dust. Consider the space dust to be at rest (not moving) before the ship hits it. Assume that all the dust that gets hit by the ship sticks to the ship and effectively increases the mass of the ship, and slows down the ship. All the motion is in one dimension.

6.1 Differential Equation

Let m be the mass of the ship at a given time t. Note that m is increasing in with t. Let v be the speed of ship at a given time t. v is decreasing with t.

By using conservation of momentum (or some other equivalent to Newton's 2nd law), find the "rocket-like" equation that relates $\mathrm{d}v$, $\mathrm{d}m$, m, and v, and solve for v as a function of m.

6.1 solution



The system of interest is the ship and all the dust that it will hit and pick up. The figure above shows the ship at time t and the ship at small time later, $t+\mathrm{d}t$, when it picks up mass $\mathrm{d}m$. All the other dust is not moving, or has already been collected by the ship. We calculate the change in momentum for the two times, $\mathrm{d}p$, giving

$$\mathrm{d}p = p(t+\mathrm{d}t) - p(t) = (m+\mathrm{d}m)\,(v+\mathrm{d}v) - mv = mv + m\mathrm{d}v + v\mathrm{d}m + \mathrm{d}v\,\mathrm{d}m - mv = m\mathrm{d}v + v\mathrm{d}m + \mathrm{d}v\,\mathrm{d}m\,. \tag{6.1}$$

There external force of this system so $\mathrm{d}p = 0$ and so

$$\boxed{m\mathrm{d}v + v\mathrm{d}m = 0} \tag{6.2}$$

where we drop the higher order (very small) $\mathrm{d}v\,\mathrm{d}m$ term. We can solve for m in terms of v like so

$$\int \frac{\mathrm{d}v}{v} = -\int \frac{\mathrm{d}m}{m}\,. \tag{6.3}$$

We can use the condition that at some time $m = m_0$ and $v = v_0$ so that

$$\int_{v'=v_0}^{v} \frac{\mathrm{d}v'}{v'} = -\int_{m'=m_0}^{m} \frac{\mathrm{d}m'}{m'} \quad\Rightarrow\quad \ln\frac{v}{v_0} = -\ln\frac{m}{m_0} \quad\Rightarrow\quad \ln\frac{v}{v_0} = \ln\frac{m_0}{m} \quad\Rightarrow\quad \frac{v}{v_0} = \frac{m_0}{m} \quad\Rightarrow\quad mv = m_0 v_0\,. \tag{6.4}$$

6.2 Find $\frac{\mathrm{dm}}{\mathrm{dt}}$

A is the cross-sectional area of the ship that is passing (cutting) through the dust. Find $\frac{\mathrm{d}m}{\mathrm{d}t}$ as a function of v, ρ, and A using the fact that the ship adds the mass of all the dust that the ship hits.

6.2 solution

As the ship moves at speed v

$$\mathrm{d}m = \rho\,\mathrm{d}V = \rho A v\,\mathrm{d}t \quad \Rightarrow \quad \boxed{\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A v}\,. \tag{6.5}$$

6.3 Solve for m(t) and v(t)

The initial ship speed (at time $t = 0$) is v_0 and the initial ship mass is m_0. Using this with previous results, solve for $m(t)$ and $v(t)$ in terms of t, v_0, m_0, A and ρ.

6.3 solution

From above we have two dependent variables, m and v, and two equations of motion. From above, we can express these equations of motion as the following set of two equations

$$mv = m_0 v_0 \qquad \text{and} \qquad \frac{\mathrm{d}m}{\mathrm{d}t} = \rho A v \tag{6.6}$$

where we pick m_0 and v_0 to be the initial values of m and v. We can combine these two equations to get an equation for just m giving

$$\frac{\mathrm{d}m}{\mathrm{d}t} = \rho A \frac{m_0 v_0}{m} \quad \Rightarrow \quad m\mathrm{d}m = \rho A m_0 v_0 \mathrm{d}t \quad \Rightarrow \quad \int_{m'=m_0}^{m} m'\mathrm{d}m' = \rho A m_0 v_0 \int_{t'=0}^{t} \mathrm{d}t'$$

$$\Rightarrow \quad \frac{1}{2}\left(m^2 - m_0^2\right) = \rho A m_0 v_0 t \quad \Rightarrow \quad \boxed{m(t) = \sqrt{m_0^2 + 2\rho A m_0 v_0 t}}\,. \tag{6.7}$$

From this we can plug $m(t)$ into the first equation in 6.6 giving

$$\sqrt{m_0^2 + 2\rho A m_0 v_0 t}\, v = m_0 v_0 \quad \Rightarrow \quad \boxed{v(t) = \frac{v_0}{\sqrt{1 + 2\rho A \frac{v_0}{m_0} t}}}\,. \tag{6.8}$$

For those of you who must do things the hard way we can find a different differential equation for $v(t)$, which has the m dependence removed. We can start with equation 6.2 and expand the differentials as functions of time giving

$$m\mathrm{d}v + v\mathrm{d}m = 0 \quad \Rightarrow \quad m\frac{\mathrm{d}v}{\mathrm{d}t}\mathrm{d}t + v\frac{\mathrm{d}m}{\mathrm{d}t}\mathrm{d}t = 0 \quad \Rightarrow \quad m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = 0 \quad \Rightarrow \quad m\dot{v} + v\dot{m} = 0\,.$$

We can remove the $\dot{m}$ with equation 6.5 and the m from equation 6.4 giving

$$\left(\frac{m_0 v_0}{v}\right)\dot{v} + v\left(\rho A v\right) = 0 \quad \Rightarrow \quad \dot{v} = -\frac{\rho A}{m_0 v_0} v^3 \tag{6.9}$$

which can be solved to give the same result as equation 6.8.