Here are some lecture notes on scaling physical models. This extends, just a little, material that is presented in the text, so it uses, in as much as possible, the same parameter and variable symbols (notation) as in the text, but since this extends the treatment in the text there are new unconventional symbols introduced. Don't let them scare you. There is no need to remember them. They are just dummies there to help present ideas. Think beyond the symbols.

Scaling the Driven Simple Harmonic Oscillator [1]

A sinusoidally driven damped simple harmonic oscillator can be modeled by the following ordinary differential equation (ODE)

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + kx = F_0\cos\omega t, \tag{0.1}$$

where m is mass, k is a linear spring constant, b is a linear damping constant, F_0 is the force amplitude of the driving force, and ω is the driving force angular frequency. So there are five physical parameters, but we know that we do not need to study a five-dimensional parameter space in order to study this system. We know that we can divide this equation by m, giving the following equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = \frac{F_0}{m}\cos\omega t, \tag{0.2}$$

which we can rewrite as

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 2\beta\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = A\cos\omega t, \tag{0.3}$$

where $\omega_0^2 \equiv \frac{k}{m}$, $2\beta \equiv \frac{b}{m}$, and $A \equiv \frac{F_0}{m}$ are derived parameters. We have reduced a five-dimensional parameter space to a four-dimensional parameter space, by introducing new derived parameters. A distinction between the system in equation 0.1 and that in the scaled equation 0.3 is that the scaled system in equation 0.3 cannot represent the unscaled system in equation 0.1 when we have zero mass, $m = 0$, but otherwise the results gotten from equation 0.3 can be easily transformed to our original "real" physical model in equation 0.1.

We continue the scaling process started here (and in your text [2]) by introducing the change of variables $\eta = \frac{x}{x_0}$, and $\tau = \frac{t}{t_0}$, where x_0 and t_0 are yet to be determined functions of the four remaining parameters. Note that, if x_0 has units of length and t_0 has units of time, then η and τ will be dimensionless. We will find x_0 and t_0 such that there are only two parameters left in this scaled differential equation for the new variable η in terms of derivatives with respect to the new independent variable τ, like so

$$\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\nu\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \sigma\eta = \cos\tau, \tag{0.4}$$

and we'll find the two remaining derived parameters ν and σ in terms of the five original physical parameters.

Eliminating x and t from equation 0.3 in favor of η and τ gives

$$\frac{x_0}{t_0^2}\frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\beta\frac{x_0}{t_0}\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \omega_0^2 x_0\eta = A\cos(\omega t_0\tau) \;\Rightarrow\; \frac{\mathrm{d}^2\eta}{\mathrm{d}\tau^2} + 2\beta t_0\frac{\mathrm{d}\eta}{\mathrm{d}\tau} + \omega_0^2 t_0^2\eta = \frac{t_0^2}{x_0}A\cos(\omega t_0\tau) \tag{0.5}$$

where we have used $t = t_0\tau$, $x = x_0\eta$, $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{1}{t_0}\frac{\mathrm{d}}{\mathrm{d}\tau}$, and $\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{1}{t_0^2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}$ which comes from the definitions given for variables η and τ. Comparing equation 0.4 with the right most equation in 0.5 we get

$$2\beta t_0 = 2\nu \tag{0.6}$$

$$\omega_0^2 t_0^2 = \sigma \tag{0.7}$$

$$\frac{t_0^2}{x_0}A = 1 \tag{0.8}$$

$$\omega t_0 = 1. \tag{0.9}$$

Solving equations 0.8 through 0.9 gives

$$x_0 = \frac{A}{\omega^2} \tag{0.10}$$

$$t_0 = \frac{1}{\omega} \tag{0.11}$$

$$\nu = \frac{\beta}{\omega} \tag{0.12}$$

$$\sigma = \frac{\omega_0^2}{\omega^2}. \tag{0.13}$$

Writing this in terms of the original parameters gives the following length scale, x_0, time scale, τ, and two dimensionless parameters ν and σ as

$$x_0 = \frac{F_0}{m\omega^2} \qquad t_0 = \frac{1}{\omega} \qquad \nu = \frac{b}{2m\omega} \qquad \sigma = \frac{k}{m}\frac{1}{\omega^2}. \tag{0.14}$$

What's the Big Deal?

We reduced the complexity of our model. This shows that only two parameters are needed to study the full range of dynamics of this physical system (except for singular cases, like for example when mass is zero, $m = 0$). This kind of analysis is indispensable when you are using computer simulations to study the full range of the dynamics of your system as it varies with the physical parameters in your system. Using this analysis the number of parameters is reduced and therefore, so is the amount of computation needed. Of course this analysis can be applied to many other types of models, not just a driven damped simple harmonic oscillator.

References

[1] Steven H. Strogatz. *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering*, chapter 3, pages 64–66. Westview Press or Addison Wesley, Cambridge, Massachusetts, USA or New York, N.Y. USA, 1994. ISBN 0738204536, This is an undergraduate text on nonlinear dynamics used in courses at Virginia Tech. This section has an example of dimensional analysis and scaling of a second order ODE which represents an oscillation of a bead on a rotating hoop with linear damping. We'll see a system similar to this later in this course.

[2] Steve Thorton and Jerry Marion. *Classical Dynamics of Particles and Systems*, chapter 3, page 118. Brooks/Cole–Thomson Learning, Belmount, CA, USA, 2004. ISBN 0534408966. Presents a much less extensive scaling of the driven damped simple harmonic oscillator.