1 Conservation of Linear Momentum

You must admit that Lagrangian dynamics is pretty magical. Equations of motion come from just taking derivatives of a Lagrangian. In many cases, we can get equations of motion with much less work than by using Newtonian dynamics.

In Lagrangian dynamics we don't need to refer to forces, they have been replaced by the corresponding energies in the Lagrangian, $L = T - U$. In Newtonian dynamics we found that if the net external force acting on a system was zero then the total linear momentum is a constant in time. As we will see in Lagrangian dynamics we don't need to refer to forces to determine when linear momentum is a constant in time. We will explore a more Lagrangian-like cause of conservation of linear momentum, that can be extended to use in quantum mechanics, for example.

It must be kept in mind that the results that come from Newtonian dynamics must be equivalent to results gotten from Lagrangian Dynamics. So the only thing that is new is the method of approach, or the way we think about things.

Review: Conservation of Linear Momentum with the Newtonian Approach

We consider a system of n particles that interact with each other. The position of each particle is given by $\vec{r}_\alpha$, where α is an index from 1 to n. The mass of each particle is the constant m_α. So from Newton's second law the equation of motion for each particle can be written as

$$\vec{F}_\alpha = \vec{F}_{\mathrm{ext}\,\alpha} + \vec{F}_{\mathrm{in}\,\alpha} = m_\alpha \ddot{\vec{r}}_\alpha, \tag{1.1}$$

where $\vec{F}_\alpha$ is the net force acting of the particle with index α, $\vec{F}_{\mathrm{in}\,\alpha}$ is the net force coming from other particles in the system, that is the net internal force acting on the particle with index α, and $\vec{F}_{\mathrm{ext}\,\alpha}$ is the net force from forces that are not part of the system particles, or external forces that act on the particle with index α. We define $\vec{p}_\alpha \equiv m_\alpha \dot{\vec{r}}$ giving

$$\vec{F}_{\mathrm{ext}\,\alpha} + \vec{F}_{\mathrm{in}\,\alpha} = \frac{\mathrm{d}\vec{p}_\alpha}{\mathrm{d}t}. \tag{1.2}$$

We can add all n equations together to give

$$\sum_{\alpha=1}^{n} \left(\vec{F}_{\mathrm{ext}\,\alpha} + \vec{F}_{\mathrm{in}\,\alpha} \right) = \sum_{\alpha=1}^{n} \frac{\mathrm{d}\vec{p}_\alpha}{\mathrm{d}t} \quad \Rightarrow \quad \sum_{\alpha=1}^{n} \vec{F}_{\mathrm{ext}\,\alpha} + \sum_{\alpha=1}^{n} \vec{F}_{\mathrm{in}\,\alpha} = \sum_{\alpha=1}^{n} \frac{\mathrm{d}\vec{p}_\alpha}{\mathrm{d}t}\,. \tag{1.3}$$

From Newton's third law, the sum of all internal forces will pair wise add to zero. Therefore

$$\sum_{\alpha=1}^{n} \vec{F}_{\mathrm{in}\,\alpha} = 0\,. \tag{1.4}$$

So with this we see that if the sum of the net external forces is zero, $\sum_{\alpha=1}^{n} \vec{F}_{\mathrm{ext}\,\alpha} = 0$, then

$$\sum_{\alpha=1}^{n} \frac{\mathrm{d}\vec{p}_\alpha}{\mathrm{d}t} = 0 \quad \Rightarrow \quad \sum_{\alpha=1}^{n} \int \frac{\mathrm{d}\vec{p}_\alpha}{\mathrm{d}t}\,\mathrm{d}t = 0 \tag{1.5}$$

$$\Rightarrow \quad \sum_{\alpha=1}^{n} \int \mathrm{d}\vec{p}_\alpha = 0 \quad \Rightarrow \quad \sum_{\alpha=1}^{n} (\vec{p}_\alpha + \vec{c}_\alpha) = 0 \quad \Rightarrow \quad \sum_{\alpha=1}^{n} \vec{p}_\alpha = -\sum_{\alpha=1}^{n} \vec{c}_\alpha\,, \tag{1.6}$$

where $\vec{c}_\alpha$ are constants of integration. We see $\sum_{\alpha=1}^{n} \vec{p}_\alpha$ is a constant which we can call the total linear momentum in the system. $-\sum_{\alpha=1}^{n} \vec{c}_\alpha$ is a constant, which we can relabel as $\vec{p}_{\mathrm{total}}$. So with the initial premise that the net external force on a system of particles is zero, we get that the total linear momentum is constant for all time.

A more general result can be obtained by setting just one vector component of the net external force to zero to give just the corresponding component linear momentum being a constant for all time. This could be then applied multiple times to include the cases for any number of vector component directions including the case of all components, as we just showed.

Conservation of Linear Momentum with the Lagrangian Approach

In the Lagrangian formulation of classical mechanics we have replaced force as a basic tool with energies in the Lagrangian. As we will show we do not have to transform our Lagrangian formulation back to looking at forces to come up with a principle like conservation of linear momentum. Conservation of linear momentum can come about from the linear translational symmetry of the Lagrangian.

We consider a Lagrangian for n particles with generalized coordinate positions that are in Cartesian coordinates with all coordinate directions for the particle positions being the same for all the particles. A specific example would be to have all particle positions measured from the same coordinate system. The general case that we consider here just requires that particle positions be measured with coordinate systems that have the same x, y, and z directions, but not necessarily the same origin. With this requirement we may add a constant displacement to all particles by using any of the particle coordinate systems.

We will show that if a Lagrangian does not change when we add a constant infinitesimal displacement in an any direction, then the linear moment will be a constant in time.

We set the change in the Lagrangian, L, do to a infinitesimal displacement $\delta\vec{r}$ being added to all particle positions equal to zero, like so

$$\delta L \equiv L\left(\vec{r}_\alpha + \delta\vec{r}, \dot{\vec{r}}_\alpha, t\right) - L\left(\vec{r}_\alpha, \dot{\vec{r}}_\alpha, t\right) = 0\,, \tag{1.7}$$

where $\vec{r}_\alpha$ is the position of the α-th particle, $\delta\vec{r}$ is the infinitesimal displacement that we are adding to all the particle positions, and $\dot{\vec{r}}_\alpha$ is the velocity of the α-th particle. L depends on all particle positions and so the $\vec{r}_\alpha$ is representative of the dependence of L on all n particles where α, the particle index, goes from 1 to n. The same can be said for the particle velocity, $\dot{\vec{r}}_\alpha$, dependence.

The next step we will show that for our particular variation of L we have

$$\delta L = \sum_{\alpha,i} \frac{\partial L}{\partial x_{\alpha,i}}\,\delta x_i = 0\,, \tag{1.8}$$

where $x_{\alpha i}$ is the i-th Cartesian component of the α-th particle, i goes from 1 to 3, α goes from 1 to n, and δx_i is the is the i-th Cartesian component of $\delta\vec{r}$.

Digression: Partial Derivatives

Equation 1.8 is the first order, in $\delta\vec{r}$, expansion of δL. Since $\delta\vec{r}$ is infinitesimal this is exact. This is simular to the first order terms in a multi-variable Taylor series expansion of a multi-variable function, but since the same constant $\delta\vec{r}$ is added to each vector variable, it is not quite the same.

We will use the following as an operating definition for a partial derivative

$$f(x+\epsilon, y) - f(x, y) = \frac{\partial f}{\partial x}\,\epsilon, \tag{1.9}$$

where f is a function of the two variables x and y, and ϵ is infinitesimal. We can expand this idea a little with the following

$$\begin{aligned} &f(x+\epsilon, y+\epsilon) - f(x,y) = f(x+\epsilon, y+\epsilon) - f(x, y+\epsilon) + f(x, y+\epsilon) - f(x,y) \\ &= \left(\left.\frac{\partial f}{\partial x}\right|_{x, y=y+\epsilon}\right)\epsilon + \frac{\partial f}{\partial y}\,\epsilon = \left(\frac{\partial f}{\partial x} + \frac{\partial \frac{\partial f}{\partial x}}{\partial y}\,\epsilon\right)\epsilon + \frac{\partial f}{\partial y}\,\epsilon = \frac{\partial f}{\partial x}\,\epsilon + \frac{\partial f}{\partial y}\,\epsilon, \end{aligned} \tag{1.10}$$

where all functions are evaluated at x and y unless otherwise noted. Note that in the last step we drop the term that is of size of order ϵ^2. The last equal sign is valid since ϵ is infinitesimal, and when comparing infinitesimal numbers only the lowest power terms of the infinitesimal number needs to be kept. Next we will extend and apply this idea to δL.

To validate equation 1.8 can expand the all the position variables into components in L in equation 1.7 giving

$$
\begin{aligned}
\delta L & \\
= \quad & L\left(x_{1,1}+\delta x_1, x_{1,2}+\delta x_2, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}, x_{n,3}, \dot{\vec{r}}_\alpha, t\right) \\
= \quad & L\left(x_{1,1}+\delta x_1, x_{1,2}+\delta x_2, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}+\delta x_2, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}+\delta x_2, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}+\delta x_3, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}+\delta x_1, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}+\delta x_2, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}+\delta x_3, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}+\delta x_1, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
& . \\
& . \\
& . \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}+\delta x_2, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
+ \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}, x_{n,3}+\delta x_3, \dot{\vec{r}}_\alpha, t\right) \\
- \quad & L\left(x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}, ..., x_{n,1}, x_{n,2}, x_{n,3}, \dot{\vec{r}}_\alpha, t\right) \\
& = 0,
\end{aligned}
\tag{1.11}
$$

where we have added and subtracted L evaluated at a sequence of position values. All the L functions in this expressions add to zero in pairs, except for the first and the last. We see that this is equal to

$$\begin{aligned}
\delta L = & \quad \frac{\partial L}{\partial x_{1,1}}\,\delta x_1 + \frac{\partial L}{\partial x_{1,2}}\,\delta x_2 + \frac{\partial L}{\partial x_{1,3}}\,\delta x_3 \\
& + \quad \frac{\partial L}{\partial x_{2,1}}\,\delta x_1 + \frac{\partial L}{\partial x_{2,2}}\,\delta x_2 + \frac{\partial L}{\partial x_{2,3}}\,\delta x_3 \\
& \quad . \\
& \quad . \\
& \quad . \\
& + \quad \frac{\partial L}{\partial x_{n,1}}\,\delta x_1 + \frac{\partial L}{\partial x_{n,2}}\,\delta x_2 + \frac{\partial L}{\partial x_{n,3}}\,\delta x_3 \\
= & \quad 0\,,
\end{aligned} \tag{1.12}$$

which is equation 1.8

Continuing from Equation 1.8

Since each component of infinitesimal displacements, δx_i, are independent, we have from equation 1.8

$$\sum_\alpha \frac{\partial L}{\partial x_{\alpha i}} = 0\,, \tag{1.13}$$

this with Lagrange's equations,

$$\frac{\partial L}{\partial x_{\alpha,i}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_{\alpha,i}} = 0 \tag{1.14}$$

gives

$$\sum_\alpha \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_{\alpha,i}} = 0 \quad \Rightarrow \quad \sum_\alpha \frac{\partial L}{\partial \dot{x}_{\alpha,i}} = \text{constant}\,. \tag{1.15}$$

So we see that each component ($i = 1, 2, 3$) of the total linear momentum is a constant. This we got from the premise that the Lagrangian did not change when all the particles in the Lagrange are displaced an infinitesimal amount. So we have shown that when the Lagrangian is invariant under an infinitesimal translation then linear momentum is conserved. Clearly if the Lagrangian is invariant under a finite translation then it will also be invariant under an infinitesimal translation. So linear momentum is conserved for a system with a Lagrangian that is invariant under a finite as well as infinitesimal translation. The analysis of the infinitesimal translation is easier than the finite translation.