1 From Lagrangian Dynamics to Hamiltonian Dynamics

Here we present a straight forward, but not so elegant, approach to deriving the Hamilton's equations of motion from Lagrange's equations of motion. This method starts with Lagrange's equations of motion, makes a change of variables, and in the process shows that it is convenient to define the Hamiltonian, though defining the Hamiltonian is not required. This is the same way that defining the Lagrangian is not required when deriving Lagrange's equations of motion from Newton's equations of motion, though not defining the Lagrangian, $L \equiv T - U$, would make things more difficult. The advantage of this method is that it does not mysteriously introduce the Hamiltonian function as a starting point.

We describe a mechanical system by s generalized coordinates q_i and generalized velocities $\dot{q}_i$, with i going from 1 to s. The Lagrangian

$$L\left(q_1, q_2, q_3, ..., q_s, \dot{q}_1, \dot{q}_2, \dot{q}_3, ..., \dot{q}_s, t\right), \tag{1.1}$$

we write in short-hand form as $L\left(q_i, \dot{q}_i, t\right)$.

Lagrange's equations of motion are

$$\frac{\partial L}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} = 0\,, \tag{1.2}$$

where i is from 1 to s.

In Hamiltonian dynamics, we make a change of variables from q_i, $\dot{q}_i$, and t to q_i, p_i, and t, where

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}\,, \tag{1.3}$$

which we call the generalized momentum conjugate to q_i, where again i goes from 1 to s. When using Lagrangian dynamics, for each generalized coordinates q_i there is one second order ordinary differential equation. For Hamiltonian dynamics, there will be one first order ordinary differential equation for each q_i, and one for each p_i. These Hamilton's equations of motion must be equivalent to Lagrange's equations of motion.

We derive Hamilton's equations from Lagrange's equations in what follows. We can rewrite Lagrange's equations as

$$\frac{\mathrm{d}}{\mathrm{d}t}p_i = \frac{\partial L}{\partial q_i} \quad \Rightarrow \quad \dot{p}_i = \frac{\partial L}{\partial q_i}\,, \tag{1.4}$$

where the partial derivative of L is taken while holding q_k $(k \neq i)$, $\dot{q}_i$ $(i = 1, 2, ..., s)$ and t constant. To make the change of variables complete we need to put L as a function of q_i, p_i, $(i = 1, 2, ..., s)$ and t. We must assume that $\dot{q}_i$ can be written as a function of q_k, p_k, $(k = 1, 2, ..., s)$ and t. So

$$\dot{q}_i = \dot{q}_i\left(q_1, q_2, q_3, ..., q_s, \dot{q}_1, \dot{q}_2, \dot{q}_3, ..., \dot{q}_s, t\right) = \dot{q}_i\left(q_k, p_k, t\right)\,, \tag{1.5}$$

where $k = 1, 2, ..., s$, and we introduced the same short-hand notation for variable dependence. L can be made as a function of q_i, p_i, and t $(i = 1, 2, ..., s)$ like so

$$L\left(q_i, \dot{q}_i, t\right) = L'\left(q_i, \dot{q}_i\left(q_k, p_k, t\right), t\right)\,. \tag{1.6}$$

We refer to L' as the Lagrangian when it is written as a function of q_i, p_i, and t $(i = 1, 2, ..., s)$, and L as the same Lagrangian when it is written as a function of q_i, $\dot{q}_i$, and t $(i = 1, 2, ..., s)$. So we replace the L equation 1.4 with L', keeping in mind that the $\frac{\partial L}{\partial q_i}$ is the partial derivative with respect to q_i holding q_k $(k \neq i)$, $\dot{q}_i$ $(i = 1, 2, ..., s)$ and t constant. We see that (here's the hardest part)

$$\frac{\partial L'}{\partial q_i} = \frac{\partial L}{\partial q_i} + \sum_{k=1}^{s}\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_i} \quad \Rightarrow \quad \frac{\partial L}{\partial q_i} = \frac{\partial L'}{\partial q_i} - \sum_{k=1}^{s}\frac{\partial L}{\partial \dot{q}_k}\frac{\partial \dot{q}_k}{\partial q_i} = \frac{\partial L'}{\partial q_i} - \sum_{k=1}^{s} p_k\frac{\partial \dot{q}_k}{\partial q_i} = \frac{\partial L'}{\partial q_i} - \frac{\partial}{\partial q_i}\sum_{k=1}^{s} p_k \dot{q}_k$$

$$\Rightarrow \quad \frac{\partial L}{\partial q_i} = -\frac{\partial}{\partial q_i}\left(\sum_{k=1}^{s} p_k \dot{q}_k - L'\right) \tag{1.7}$$

where we have used $\frac{\partial p_k}{\partial q_i} = 0$ because p_k $(k = 1, 2, ..., s)$ and q_i $(i = 1, 2, ..., s)$ are independent variables. We can define H as

$$\boxed{H\left(q_i, p_i, t\right) = \sum_{k=1}^{s} p_k \dot{q}_k - L}, \tag{1.8}$$

where it is understood that the right side of this equation must be written without any $\dot{q}_i$ $(i = 1, 2, ..., s)$ dependence.

With this and equation 1.7 we may rewrite equation 1.4 as

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q_i}}. \tag{1.9}$$

This is Hamilton's equation of motion for p_i. To get the q_i equation of motion we see

$$\frac{\partial L'}{\partial p_i} = \sum_{k=1}^{s} \frac{\partial L}{\partial \dot{q}_k} \frac{\partial \dot{q}_k}{\partial p_i} = \sum_{k=1}^{s} p_k \frac{\partial \dot{q}_k}{\partial p_i} \,. \tag{1.10}$$

We can split the sum into two parts like so

$$\frac{\partial}{\partial p_i} \sum_{k=1}^{s} p_k \dot{q}_k = \dot{q}_i + \sum_{k=1}^{s} p_k \frac{\partial \dot{q}_k}{\partial p_i} = \dot{q}_i + \frac{\partial}{\partial p_i} \left(\sum_{k=1}^{s} p_k \dot{q}_k \right) \quad \Rightarrow \quad \sum_{k=1}^{s} p_k \frac{\partial \dot{q}_k}{\partial p_i} = \frac{\partial}{\partial p_i} \left(\sum_{k=1}^{s} p_k \dot{q}_k \right) - \dot{q}_i \,. \tag{1.11}$$

and we see that we get $\dot{q}_i$ to appear. With this, equation 1.10 may be written as

$$\frac{\partial L'}{\partial p_i} = \frac{\partial}{\partial p_i} \left(\sum_{k=1}^{s} p_k \dot{q}_k \right) - \dot{q}_i \quad \Rightarrow \quad \dot{q}_i = \frac{\partial}{\partial p_i} \left(\sum_{k=1}^{s} p_k \dot{q}_k \right) - \frac{\partial L'}{\partial p_i} = \frac{\partial}{\partial p_i} \left(\sum_{k=1}^{s} p_k \dot{q}_k - L' \right) \,. \tag{1.12}$$

We rewrite this with the definition of H from equation 1.8 to get

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}}. \tag{1.13}$$

This is Hamilton's equation of motion for q_i.

So now we have taking Largrange's equation of motion, n second order differential equation for q_i, and with a change of variables made it into $2n$ first order differential equations for p_i and q_i, called Hamilton's equations of motion.