1 Kepler Orbits

We present some of the details of how to calculate the orbital path in polar coordinates r, and θ, as $r(\theta)$, from the differential equations of motion. We show two ways to do this: **(section 1.1)** we start with the conservation of energy in the reduced mass system which is the first order differential equations of motion for $r(t)$, and **(section 1.2)** by solving the second order differential equations of motion for $r(t)$. In both cases we replace t dependence of r with θ dependence. Clearly, in both cases we will get the same result. Both methods are equivalent. We leave deriving the equations of motion and the total energy of this system to previous presentations.

We start with two particles with masses m_1 and m_2 and positions $\vec{r}_1$ and $\vec{r}_2$. The to particles interact with each other with a force that has an associated potential energy $U(|\vec{r}_1 - \vec{r}_2|)$. We leave it as an exercise to the reader to show that the relative motion $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ evolves independently of the center of mass motion, $\vec{R} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$. Because angular momentum about the center of mass is constant, the motion of $\vec{r}$ must be in a plane. So we can describe the motion of $\vec{r}$ in this plane in polar coordinates with variables r and θ.

When we refer to the system we are referring to the motion of $\vec{r}$, and we are ignoring the motion of the center of mass, $\vec{R}$.

1.1 Kepler Orbits from Integrating the Energy Equation

The total energy of this system, E, is a constant. We leave it as an exercise to the reader to show that (we already showed this)

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 - G\frac{m_1 m_2}{r} = \frac{1}{2}\mu\dot{r}^2 + \frac{l^2}{2\mu r^2} - \frac{k}{r} \tag{1.1}$$

where $k \equiv Gm_1m_2$ and we have used the θ equation of motion $\mu r^2\dot{\theta} = l$, where l is the angular momentum. We may solve for $\dot{r}$ in equation 1.1 giving

$$\dot{r} = \sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}\,. \tag{1.2}$$

Integrating this would give us $r(t)$, but we want $r(\theta)$. We can change the independent variable using

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\left(\frac{l}{\mu r^2}\right) = \frac{l}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\,. \tag{1.3}$$

Putting this together with equation 1.2 gives

$$\frac{l}{\mu r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} = \sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)} \quad\Rightarrow\quad \mathrm{d}\theta = \frac{\frac{l}{\mu r^2}\,\mathrm{d}r}{\sqrt{\frac{2}{\mu}\left(E - \frac{l^2}{2\mu r^2} + \frac{k}{r}\right)}} \quad\Rightarrow\quad \theta = \frac{l}{\sqrt{2\mu}}\int\frac{\frac{\mathrm{d}r}{r^2}}{\sqrt{E - \frac{l^2}{2\mu r^2} + \frac{k}{r}}} \tag{1.4}$$

We make a change of variables $u = 1/r$, so $\mathrm{d}u = -\frac{\mathrm{d}r}{r^2}$ and we get

$$\theta = \frac{l}{\sqrt{2\mu}}\int\frac{-\mathrm{d}u}{\sqrt{E - \frac{l^2}{2\mu}u^2 + ku}} = -\frac{l}{\sqrt{2\mu}}\int\frac{\mathrm{d}u}{\sqrt{\frac{l^2}{2\mu}}\sqrt{\frac{2E\mu}{l^2} + \frac{2k\mu}{l^2}u - u^2}} = -\int\frac{\mathrm{d}u}{\sqrt{\frac{2E\mu}{l^2} + \frac{2k\mu}{l^2}u - u^2}}\,. \tag{1.5}$$

Now we complete the square of the polynomial in u giving

$$\theta = -\int\frac{\mathrm{d}u}{\sqrt{\frac{2E\mu}{l^2} + \left(\frac{k\mu}{l^2}\right)^2 - \left(\frac{k\mu}{l^2}\right)^2 + \frac{2k\mu}{l^2}u - u^2}} = -\int\frac{\mathrm{d}u}{\sqrt{\frac{2E\mu}{l^2} + \left(\frac{k\mu}{l^2}\right)^2 - \left(u - \frac{k\mu}{l^2}\right)^2}}\,. \tag{1.6}$$

We make another change of variables $x = u - \frac{k\mu}{l^2}$, so $\mathrm{d}u = \mathrm{d}x$. This gives

$$\theta = -\int\frac{\mathrm{d}x}{\sqrt{\frac{2E\mu}{l^2} + \left(\frac{k\mu}{l^2}\right)^2 - x^2}}\,.$$

Our integral is now of the form

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = -\cos^{-1}\frac{x}{|a|}\,, \tag{1.7}$$

where a is a constant. Using this we get

$$\theta = \cos^{-1}\left(\frac{x}{\sqrt{\frac{2E\mu}{l^2} + \left(\frac{k\mu}{l^2}\right)^2}}\right) + c_1 = \cos^{-1}\left(\frac{\frac{1}{r} - \frac{k\mu}{l^2}}{\sqrt{\left(\frac{k\mu}{l^2}\right)^2 + \frac{2E\mu}{l^2}}}\right) + c_1\,, \tag{1.8}$$

where c_1 is an integration constant. Rearranging gives

$$\Rightarrow \quad \cos(\theta - c_1) = \frac{\frac{1}{r} - \frac{k\mu}{l^2}}{\sqrt{\left(\frac{k\mu}{l^2}\right)^2 + \frac{2E\mu}{l^2}}} \quad \Rightarrow \quad \frac{1}{r} - \frac{k\mu}{l^2} = \sqrt{\left(\frac{k\mu}{l^2}\right)^2 + \frac{2E\mu}{l^2}}\cos(\theta - c_1)$$

$$\Rightarrow \quad \frac{1}{r} = \frac{k\mu}{l^2} + \sqrt{\left(\frac{k\mu}{l^2}\right)^2 + \frac{2E\mu}{l^2}}\cos(\theta - c_1) \quad \Rightarrow \quad \frac{\left(\frac{l^2}{k\mu}\right)}{r} = 1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos(\theta - c_1)\,. \tag{1.9}$$

Choosing the value of θ to be zero when r is at a minimum value, makes $c_1 = 0$ giving

$$\boxed{\frac{\left(\frac{l^2}{k\mu}\right)}{r} = 1 + \sqrt{1 + \frac{2El^2}{\mu k^2}}\cos\theta}\,. \tag{1.10}$$

We can define the **latus rectum**, α, and the **eccentricity**, ϵ, by

$$\boxed{\alpha \equiv \frac{l^2}{k\mu}\,, \quad \text{and} \quad \epsilon \equiv \sqrt{1 + \frac{2El^2}{\mu k^2}}}\,. \tag{1.11}$$

So we may rewrite the orbital path equation 1.10 as

$$\boxed{\frac{\alpha}{r} = 1 + \epsilon\cos\theta}\,. \tag{1.12}$$

1.2 Kepler Orbits from Second Order Differential Equation of Motion

We start with the second order ordinary differential equation of motion for r which is

$$\mu\ddot{r} = \mu r\dot{\theta}^2 - \frac{k}{r^2} = \frac{l^2}{\mu r^3} - \frac{k}{r^2}\,, \tag{1.13}$$

where $k \equiv Gm_1m_2$ and we have used the θ equation of motion $\mu r^2\dot{\theta} = l$, where l is the angular momentum. Integrating this would give us $r(t)$, but we want $r(\theta)$. We can change the independent variable by using equation 1.3. Since this is shown in your text in detail we present, without proof, the resulting second order ordinary differential equation for $r(\theta)$

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{\mu r^2}{l^2}F(r) = \frac{\mu r^2}{l^2}\frac{k}{r^2}\,, \tag{1.14}$$

where $F(r) \equiv -\frac{\partial U(r)}{\partial r}$, and in our case we have the gravitational force $F(r) = -\frac{k}{r^2}$. With the change of variable $u = 1/r$ we get

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = \frac{k\mu}{l^2} \quad \Rightarrow \quad \frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -\left(u - \frac{k\mu}{l^2}\right)\,, \tag{1.15}$$

which has the general solution

$$u - \frac{k\mu}{l^2} = C_2 \cos(\theta + c_1)\,, \tag{1.16}$$

where C_2 and c_1 are constants of integration. Rewriting equation 1.16 in term of r gives

$$\frac{1}{r} = \frac{k\mu}{l^2} + C_2 \cos(\theta + c_1)\,. \tag{1.17}$$

We can, as in the section 1.1, set $r(\theta = 0)$ equal to a minimum value so that $c_1 = 0$. The integration constant C_2 we can relate to the energy E. First we rewrite equation 1.17 as

$$\frac{\left(\frac{l^2}{k\mu}\right)}{r} = 1 + \epsilon \cos\theta\,, \tag{1.18}$$

where $\epsilon = \left(\frac{l^2}{k\mu}\right) C_2$ is new integration constant which we still need to find. ϵ is the eccentricity as we defined it before, but in this derivation we do not have it as a function of physical parameters yet because we did not do an integration to get this solution as we did in section 1.1. We can find ϵ as a function of E by solving for the minimum r value in equation 1.18 and comparing that to the minimum r value gotten from the conservation of energy. Of course you should get the same result as in equation 1.11.

This method of getting $r(\theta)$ may appear shorter than that in section 1.1, but we also skipped more steps in this section than in section 1.1. You can use either of the two methods presented here to do your next homework.