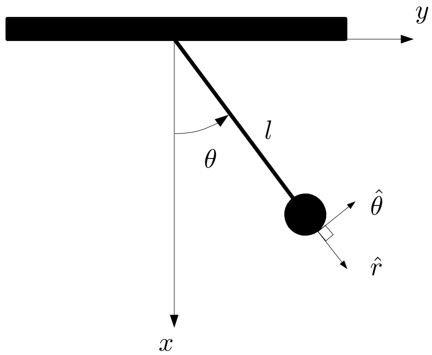
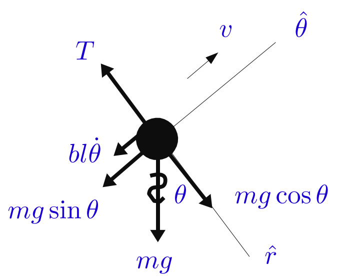
1 1-D pendulum

A point particle, with mass m, pivots without friction on the end of a massless stretch-less stick of length l. There is a linear damping force on the particle that acts in the direction opposite the motion of the particle with a force of magnitude bv, where b is a constant and v is the speed of the particle. Draw a free body diagram of the particle, and apply Newton's 2nd law to find the ordinary differential equation of motion for θ for this 1-D pendulum shown below. Express your answer in terms of g, l, m, b, and, θ and its time derivatives.



1.0 solution



From Newton's 2nd law

$$\sum F_\theta = -mg\sin\theta - bl\dot{\theta} = ma_\theta \tag{1.1}$$

$$= ml\ddot{\theta} \tag{1.2}$$

which gives

$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{b}{m}\dot{\theta}}, \tag{1.3}$$

or

$$\boxed{\ddot{\theta} + \frac{g}{l}\sin\theta + \frac{b}{m}\dot{\theta} = 0}. \tag{1.4}$$

I noticed that many students did this problem by writing the equations of motion (Newton's 2nd law) in Cartesian coordinates. It's not an easy way to do the problem. Here's one way to do it that way:

From Newton's 2nd law

$$\sum F_x = \quad mg - b\dot{x} - T\cos\theta = m\ddot{x} \tag{1.5}$$

$$\sum F_y = \quad -b\dot{y} - T\sin\theta = m\ddot{y} \tag{1.6}$$

where the x and y components of the of the linear drag force, $-b\dot{x}$ and $-b\dot{y}$, is clear to see in the equation

$$\vec{F}_{\text{drag}} \equiv -b\dot{\vec{r}} = -b\dot{x}\hat{x} - b\dot{y}\hat{y}, \tag{1.7}$$

and θ is dependent on x and y such that,

$$\tan\theta = \frac{y}{x}. \tag{1.8}$$

You must keep in mind that θ is a variable, and not a parameter like the fixed inclination angle of an inclined plane. θ can be written as a function of x and y, and we can replace $\cos\theta$ and $\sin\theta$ in equations 1.5 and 1.6 so we get

$$\sum F_x = mg - b\dot{x} - T\frac{x}{\sqrt{x^2+y^2}} = m\ddot{x} \tag{1.9}$$

$$\sum F_y = -b\dot{y} - T\frac{y}{\sqrt{x^2+y^2}} = m\ddot{y}. \tag{1.10}$$

We now have two coupled second order differential equations, and an unknow tension force T, and a constraint in the motion that can be written as

$$l^2 = x^2 + y^2 \tag{1.11}$$

where l is the length of the pendulum. Without this constrain we would not have a pendulum. The unknow tension force T can be eliminated between 1.9 and 1.10 which would leave us with one second order differential equation with two variables, x and y, and with θ no longer present. There are two coordinate degrees of freedom, x and y which are not independent. One can be eliminated using $l^2 = x^2 + y^2$ leaving one degree of freedom.

The problem states that the answer should be expressed in terms of the variable θ. We can make the change of variables from x and y, Cartesian coordinates, to r and θ, polar coordinates. The constrain equation $l^2 = x^2 + y^2$ makes r a constant (l) of the motion.

The transformation equations, with derivatives, can be written as

$$\begin{array}{ll} x = l\cos\theta & y = l\sin\theta \\ \dot{x} = -l\dot{\theta}\sin\theta & \dot{y} = l\dot{\theta}\cos\theta \\ \ddot{x} = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta & \ddot{y} = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta, \end{array} \tag{1.12}$$

where we have used the fact that $r = l$ is a constant in time.

Using this we rewrite equations 1.5 and 1.6 (the equations of motion) as

$$g + \frac{b}{m}l\dot{\theta}\sin\theta - \frac{T}{m}\cos\theta = -l\dot{\theta}^2\cos\theta - l\ddot{\theta}\sin\theta \tag{1.13}$$

$$-\frac{b}{m}l\dot{\theta}\cos\theta - \frac{T}{m}\sin\theta = -l\dot{\theta}^2\sin\theta + l\ddot{\theta}\cos\theta. \tag{1.14}$$

We can add $-\sin\theta$ times equation 1.13 to $\cos\theta$ times equation 1.14 giving

$$-g\sin\theta - \frac{b}{m}l\dot{\theta}\sin^2\theta - \frac{b}{m}l\dot{\theta}\cos^2\theta = l\ddot{\theta}\sin^2\theta + l\ddot{\theta}\cos^2\theta, \tag{1.15}$$

which we can rewrite as

$$-\frac{g}{l}\sin\theta - \frac{b}{m}\dot{\theta}\left(\sin^2\theta + \cos^2\theta\right) = \ddot{\theta}\left(\sin^2\theta + \cos^2\theta\right), \tag{1.16}$$

which we can rewrite using the trigonometric identity $\sin^2\theta + \cos^2\theta = 1$ as

$$\boxed{-\frac{g}{l}\sin\theta - \frac{b}{m}\dot{\theta} = \ddot{\theta}}. \tag{1.17}$$

This shows that the two methods give the same results, but the polar coordinates put the equation of motion into a simpler form to start with, and seems to be a more natural choice of coordinates due to the nature of the constraint of the motion ($l^2 = x^2 + y^2$), circular motion.

You might ask yourself, "why have I not seen this before in PHYS 2306? We studied pendulums in PHYS 2306". In PHYS 2306 there was usually no damping and θ was usually a small angle, so that simplified things considerably.