1 Stationary Integral

Find $y(x)$ such that the following integral is stationary,

$$J = \int_{x_1}^{x_2} \left(\frac{1}{2} \, {y'}^2 \right) \, \mathrm{d}x, \tag{1.1}$$

where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}x}$.

Hints: Do so by using the Euler equation,

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0, \tag{1.2}$$

where $f(y, y'; x) = \frac{1}{2} \, {y'}^2$. You do not have to determine the two constants of integration, just call them c_1 and c_2. Don't bother interpreting them. Don't bother interpreting J or $y(x)$. This is just an exercise.

1.0 solution

For J to be stationary Euler's equation must be satisfied. So

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x} \left(\frac{\partial f}{\partial y'} \right) = 0, \tag{1.3}$$

where

$$f(y, y'; x) = \frac{1}{2} \, {y'}^2, \tag{1.4}$$

we get

$$0 - \frac{\mathrm{d}}{\mathrm{d}x} (y') = 0 \quad \Rightarrow \quad y' = c_1 \quad \Rightarrow \quad \int \frac{\mathrm{d}y}{\mathrm{d}x} \, \mathrm{d}x = \int c_1 \, \mathrm{d}x \quad \Rightarrow \quad \boxed{y(x) = c_1 \, x + c_2} \tag{1.5}$$