1 Stationary Integral

Find $y(t)$ and $x(t)$ such that the following integral is stationary,

$$J = \int_{t_1}^{t_2} \left(\frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\dot{y}^2 \right) \mathrm{d}t, \tag{1.1}$$

where $\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t}$ and $\dot{y} \equiv \frac{\mathrm{d}y}{\mathrm{d}t}$.

Hint: Do so by using the Euler equations,

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \qquad \text{and} \qquad \frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0 \tag{1.2}$$

where $f(x,\dot{x},y,\dot{y};t) = \frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,\dot{y}^2$. You do not have to determine all the constants of integration.

1.0 solution

Plugging f into the Euler equations gives

$$0 - \frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = 0 \qquad \text{and} \qquad 0 - \frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = 0 \tag{1.3}$$

which may be integrated giving

$$\dot{x} = c_1 \qquad \text{and} \qquad \dot{y} = c_2 \qquad \Rightarrow \qquad \boxed{x = c_1 t + c_3 \qquad \text{and} \qquad y = c_2 t + c_4} \tag{1.4}$$

where c_1, c_2, c_3, and c_4 are constants of integration.