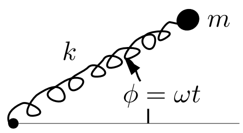
1 Spinning a Mass on a Spring

A particle, of mass m, is connected to a spring, with spring constant k. The spring has a rest length a. One end of the spring is held fixed while the particle rotates around the fixed end at a constant angular frequency ω, as shown in the figure below. The spring stays straight and connected to the particle, spinning with it about the fixed end of the spring. Let r be the position of the particle measured from the point of rotation, so the Lagrangian will be a function of r, $\dot{r}$, and maybe t, time ($L(r, \dot{r}, t)$). Using the Lagrangian method, find the equation of motion for r ($\ddot{r} = ?$).



1.0 solution

$$L = T - U = \frac{1}{2}mv^2 - \frac{1}{2}k(r-a)^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{1}{2}k(r-a)^2 = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - \frac{1}{2}k(r-a)^2 \quad (1.1)$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\omega^2 - \frac{1}{2}k(r-a)^2 . \quad (1.2)$$

Using Lagrange's equation

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} \quad \Rightarrow \quad mr\omega^2 - k(r-a) - \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = 0 \quad \Rightarrow \quad m\ddot{r} = mr\omega^2 - k(r-a) \quad (1.3)$$

$$\Rightarrow \quad \boxed{\ddot{r} = -\left(\frac{k}{m} - \omega^2\right)r + \frac{k}{m}a}. \quad (1.4)$$

We get simple harmonic motion if $\frac{k}{m} > \omega^2$. If $\frac{k}{m} < \omega^2$ we get a broken spring ($r \to \infty$ as $t \to \infty$).