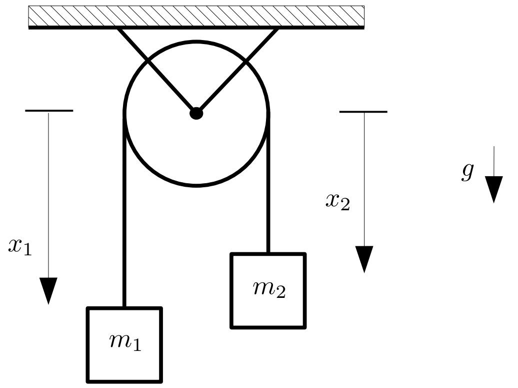
1 Atwood's Machine using a Lagrange Multiplier

An Atwood's machine machine is constructed with a fixed massless frictionless pulley, a stretch-less massless string, and two weights with masses m_1 and m_2, as shown in the figure below. Use x_1 and x_2 as the positions of hanging weights as shown.



Find the acceleration of weight 1 (the weight with mass m_1) $\ddot{x}_1$, the acceleration of weight 2 (the weight with mass m_2) $\ddot{x}_2$, and T the tension in the string, by using Lagrange's equations and a Lagrange multiplier λ.

1.0 solution

The Lagrangian is

$$L = T - U = T_1 + T_2 - U_1 - U_2 = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 - m_1 g\left(-x_1\right) - m_2 g\left(-x_2\right)$$

$$\Rightarrow \quad L\left(x_1, x_2, \dot{x}_1, \dot{x}_2\right) = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + m_1 g x_1 + m_2 g x_2. \tag{1.1}$$

The equation of constraint is

$$f\left(x_1, x_2\right) = x_1 + x_2 - l = 0 \tag{1.2}$$

where l is a constant. The set of equations of motion consist of this constraint equation, $f\left(x_1, x_2\right)$, and the two Lagrange equations

$$\frac{\partial L}{\partial x_1} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_1} + \lambda(t)\frac{\partial f}{\partial x_1} = 0 \quad \text{and} \quad \frac{\partial L}{\partial x_2} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}_2} + \lambda(t)\frac{\partial f}{\partial x_2} = 0, \tag{1.3}$$

where $\lambda(t)$ is the Lagrange multiplier, and so the generalized forces of constraint are

$$\mathcal{Q}_1 = \lambda(t)\frac{\partial f}{\partial x_1} \quad \text{and} \quad \mathcal{Q}_2 = \lambda(t)\frac{\partial f}{\partial x_2}. \tag{1.4}$$

$\mathcal{Q}_1$ will be the generalized constraint force, associated the equation of constraint, that is along the x_1 direction. Since the generalized coordinates are Cartesian the generalized constraint force $\mathcal{Q}_1$ will be a regular force. So $\mathcal{Q}_1$ will be minus the tension in the string. $\mathcal{Q}_2$ will be the generalized constraint force, associated the equation of constraint, that is along the x_2 direction. So $\mathcal{Q}_2$ will be minus the tension in the string.

Plugging L and f into equations 1.3 gives

$$m_1 g - \frac{\mathrm{d}}{\mathrm{d}t}\left(m_1\dot{x}_1\right) + \lambda\left[1\right] \quad \Rightarrow \quad m_1\ddot{x}_1 = m_1 g + \lambda \tag{1.5}$$

and

$$m_2 g - \frac{\mathrm{d}}{\mathrm{d}t}\left(m_2\dot{x}_2\right) + \lambda\left[1\right] \quad \Rightarrow \quad m_2\ddot{x}_2 = m_2 g + \lambda. \tag{1.6}$$

Differentiating the equation of constraint (equation 1.2) with respect to time (t) twice gives

$$\ddot{x}_1 + \ddot{x}_2 = 0. \tag{1.7}$$

We have three equations, 1.5, 1.5, and 1.7, and three unknowns $\ddot{x}_1$, $\ddot{x}_2$, and λ. Solving gives

$$\boxed{\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2} g}, \quad \boxed{\ddot{x}_2 = -\frac{m_1 - m_2}{m_1 + m_2} g}, \quad \text{and} \quad \boxed{\lambda = -\frac{2m_1 m_2}{m_1 + m_2}\, g}. \tag{1.8}$$

The constraint force in the string is

$$\mathcal{Q}_1 = \lambda(t)\frac{\partial f}{\partial x_1} = -\left(\frac{2m_1 m_2}{m_1 + m_2}\, g\right)(1) = -\frac{2m_1 m_2}{m_1 + m_2}\, g, \tag{1.9}$$

which is also the same as $\mathcal{Q}_2$, which is minus the tension in the string. So

$$\boxed{T = \frac{2m_1 m_2}{m_1 + m_2}\, g}. \tag{1.10}$$