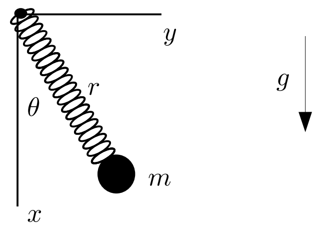
1 Springy Pendulum



The springy pendulum shown above has a rest spring length a, a spring constant k, and a bob mass of m. r is the distance from the origin (pivot point) to the bob. Find **(a)** the Lagrangian, $L(r, \theta, \dot{r}, \dot{\theta})$, and **(b)** the differential equations of motion. Simplify your results.

1.0 solution

$$L = T - U = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}k\left(r-a\right)^2 - mg\left(-x\right)$$

$$\Rightarrow \quad (\mathbf{a})\boxed{L(r,\theta,\dot{r},\dot{\theta}) = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - \frac{1}{2}k\left(r-a\right)^2 + mgr\cos\theta} \tag{1.1}$$

$$\frac{\partial L}{\partial r} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{r}} = 0 \quad \Rightarrow \quad mr\dot{\theta}^2 - k\left(r-a\right) + mg\cos\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{r}\right) = 0$$

$$\Rightarrow \quad (\mathbf{b})\boxed{m\ddot{r} = -k\left(r-a\right) + mr\dot{\theta}^2 + mg\cos\theta} \tag{1.2}$$

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} = 0 \quad \Rightarrow \quad -mgr\sin\theta - \frac{\mathrm{d}}{\mathrm{d}t}\left(mr^2\dot{\theta}\right) = 0 \quad \Rightarrow \quad -mgr\sin\theta - \left(mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}\right) = 0$$

$$\Rightarrow \quad (\mathbf{b})\boxed{r\ddot{\theta} = -g\sin\theta - 2\dot{r}\dot{\theta}} \tag{1.3}$$