1 Simple Pendulum

A simple plane pendulum has a length l, bob mass m, and is in a uniform gravitational field, g. Use θ, the angle the pendulum swings from equilibrium, as the generalized coordinate. Find **(a)** the Lagrangian $L\left(\theta,\dot{\theta}\right)$, **(b)** p_θ, the momentum conjugate to θ, as a function of m, g, l, θ and $\dot{\theta}$, **(c)** the Hamiltonian $H\left(\theta, p_\theta\right)$, **(d)** Hamilton's equation of motion for p_θ ($\dot{p}_\theta =$?), and **(e)** Hamilton's equation of motion for θ ($\dot{\theta} =$?).

1.0 solution

$$L = T - U = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta \quad \Rightarrow \quad (\mathbf{a})\ \boxed{L\left(\theta,\dot{\theta}\right) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta}. \tag{1.1}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \quad \Rightarrow \quad (\mathbf{b})\ \boxed{p_\theta = ml^2\dot{\theta}}. \tag{1.2}$$

$$H = p_\theta\dot{\theta} - L = ml^2\dot{\theta}\dot{\theta} - \left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right) \tag{1.3}$$

We have from equation 1.2

$$\dot{\theta} = \frac{p_\theta}{ml^2}. \tag{1.4}$$

So

$$H = \frac{1}{2}ml^2\left(\frac{p_\theta}{ml^2}\right)^2 = -mgl\cos\theta \quad \Rightarrow \quad (\mathbf{c})\ \boxed{H\left(\theta,p_\theta\right) = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta}. \tag{1.5}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl\sin\theta \quad \Rightarrow \quad (\mathbf{d})\ \boxed{\dot{p}_\theta = -mgl\sin\theta}. \tag{1.6}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta}\frac{p_\theta}{ml^2} \quad \Rightarrow \quad (\mathbf{e})\ \boxed{\dot{\theta} = \frac{p_\theta}{ml^2}}. \tag{1.7}$$