1 Angular Momentum

Consider two particles, one with mass m_1 and position given by $\vec{r}_1$, and the other with mass m_2 and position given by $\vec{r}_2$. $\vec{r}_1$ and $\vec{r}_2$ are both measured from the same coordinate system. The center of mass position of the two particles is,

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}\,. \tag{1.1}$$

If we measure the positions $\vec{r}_1$ and $\vec{r}_2$ from the position of the center of mass then

$$0 = m_1\vec{r}_1 + m_2\vec{r}_2\,. \tag{1.2}$$

The angular momentum of the two particles is

$$\vec{L} = m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2\,. \tag{1.3}$$

Show that the angular momentum of the two particles as measured about the center of mass position is given by

$$\vec{L} = \mu\,\vec{r} \times \dot{\vec{r}}, \quad \text{where} \quad \mu \equiv \frac{m_1 m_2}{m_1 + m_2}\,, \quad \text{and} \quad \vec{r} \equiv \vec{r}_1 - \vec{r}_2\,. \tag{1.4}$$

1.0 solution

We find r_1 and r_2 in terms of r from

$$0 = m_1\vec{r}_1 + m_2\vec{r}_2 \tag{1.5}$$

and

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \tag{1.6}$$

giving

$$\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}, \qquad \vec{r}_2 = -\frac{m_1}{m_1 + m_2}\vec{r}, \qquad \dot{\vec{r}}_1 = \frac{m_2}{m_1 + m_2}\dot{\vec{r}}, \qquad \text{and} \qquad \dot{\vec{r}}_2 = -\frac{m_1}{m_1 + m_2}\dot{\vec{r}}. \tag{1.7}$$

Plugging this into $\vec{L} = m_1\vec{r}_1 \times \dot{\vec{r}}_1 + m_2\vec{r}_2 \times \dot{\vec{r}}_2$ gives

$$\vec{L} = m_1\frac{m_2}{m_1 + m_2}\vec{r} \times \frac{m_2}{m_1 + m_2}\dot{\vec{r}} + m_2\frac{m_1}{m_1 + m_2}\vec{r} \times \frac{m_1}{m_1 + m_2}\dot{\vec{r}} = \left[m_1\frac{m_2^2}{(m_1 + m_2)^2} + m_2\frac{m_1^2}{(m_1 + m_2)^2}\right]\vec{r} \times \dot{\vec{r}}$$

$$= \left[m_1 m_2\frac{m_2 + m_1}{(m_1 + m_2)^2}\right]\vec{r} \times \dot{\vec{r}} \quad \Rightarrow \boxed{\vec{L} = \frac{m_1 m_2}{m_1 + m_2}\,\vec{r} \times \dot{\vec{r}} = \mu\,\vec{r} \times \dot{\vec{r}}}. \tag{1.8}$$