1 Closed Kepler Orbits in Cartesian Coordinates

The general solution of the orbital path of the relative position of two particles in polar coordinates (r, θ) can be written as

$$\frac{\alpha}{r} = 1 + \epsilon \cos\theta \tag{1.1}$$

where α and ϵ are constants that depend on initial conditions.

Show that equation 1.1 can be written in Cartesian coordinates, $x = r\cos\theta$ and $y = r\sin\theta$, as

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{1.2}$$

So find a, b, and d, as a function of α and ϵ.

1.0 solution

Substituting $x = r\cos\theta$ and $y = r\sin\theta$ into equation 1.1 gives

$$\alpha = r + \epsilon x \quad \Rightarrow \quad \sqrt{x^2+y^2} = \alpha - \epsilon x \quad \Rightarrow \quad x^2 + y^2 = \alpha^2 - 2\alpha\epsilon x + \epsilon^2 x^2$$

$$\Rightarrow \quad \left(1-\epsilon^2\right) x^2 + 2\alpha\epsilon x + y^2 = \alpha^2 \quad \Rightarrow \quad x^2 + 2\frac{\alpha\epsilon}{(1-\epsilon^2)}x + \frac{y^2}{(1-\epsilon^2)} = \frac{\alpha^2}{(1-\epsilon^2)} \tag{1.3}$$

We can complete the square of the x terms giving

$$x^2 + 2\frac{\alpha\epsilon}{(1-\epsilon^2)}x + \frac{\alpha^2\epsilon^2}{(1-\epsilon^2)^2} + \frac{y^2}{(1-\epsilon^2)} = \frac{\alpha^2}{(1-\epsilon^2)} + \frac{\alpha^2\epsilon^2}{(1-\epsilon^2)^2}$$

$$\Rightarrow \quad \left(x + \frac{\alpha\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{\alpha^2\left(1-\epsilon^2\right) + \alpha^2\epsilon^2}{(1-\epsilon^2)^2} \quad \Rightarrow \quad \left(x + \frac{\alpha\epsilon}{1-\epsilon^2}\right)^2 + \frac{y^2}{1-\epsilon^2} = \frac{\alpha^2}{(1-\epsilon^2)^2}$$

$$\Rightarrow \quad \boxed{\frac{\left(x + \frac{\alpha\epsilon}{1-\epsilon^2}\right)^2}{\left(\frac{\alpha}{1-\epsilon^2}\right)^2} + \frac{y^2}{\left(\frac{\alpha}{\sqrt{1-\epsilon^2}}\right)^2} = 1}\,. \tag{1.4}$$

So

$$\boxed{a = \frac{\alpha}{1-\epsilon^2}\,, \quad b = \frac{\alpha}{\sqrt{1-\epsilon^2}}\,, \quad \text{and} \quad d = \frac{\alpha\epsilon}{1-\epsilon^2}}\,. \tag{1.5}$$