1 Derive the Quadratic Formula

By the method of Completing the Square, solve for x in the following equation

$$ax^2 + bx + c = 0\,, \tag{1.1}$$

where a, b, and c are positive constants.

1.0 solution

$$ax^2 + bx + c = 0 \quad \Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{(2a)^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{(2a)^2} - \frac{c}{a} \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{(2a)^2} \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right) = \pm\sqrt{\frac{b^2 - 4ac}{(2a)^2}}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}\,. \tag{1.2}$$