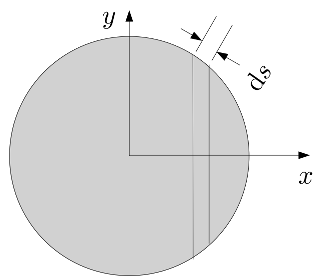
1 Moment of Inertia of a Hollow Sphere

Find the moment of inertia of a uniform thin-walled hollow sphere of radius R and total mass M as it rotates about an axis through the center of the sphere. Getting started: The figure below uses the x-axis as the axis of rotation. $I = \int_{x=-R}^{R} y^2\,\mathrm{d}m = \int_{x=-R}^{R} y^2\,(\sigma\,2\pi y\,\mathrm{d}s)$ where σ is the mass per unit area for the sphere, and $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$.



1.0 solution

We get the moment of inertia about the x-axis by adding the moment of inertia many rings with radius $y = \sqrt{R^2 - x^2}$

$$I = \int_{x=R}^{-R} y^2\,\mathrm{d}m = \int_{x=R}^{-R} y^2\,(\sigma\,2\pi y\,\mathrm{d}s) \tag{1.1}$$

where $\sigma = \frac{M}{4\pi R^2}$ is the mass per unit area of the sphere and

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \tag{1.2}$$

$$R^2 = x^2 + y^2 \quad \Rightarrow \quad 0 = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} \quad \Rightarrow \quad \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{x^2}{y^2}\,. \tag{1.3}$$

So

$$\mathrm{d}s = \sqrt{1 + \frac{x^2}{y^2}}\,\mathrm{d}x = \sqrt{\frac{y^2 + x^2}{y^2}}\,\mathrm{d}x = \frac{R}{y}\mathrm{d}x\,. \tag{1.4}$$

So with this and equation 1.1

$$I = \int_{x=-R}^{R} y^2 \left(\frac{M}{4\pi R^2}\right) 2\pi y\,\frac{R}{y}\mathrm{d}x = \frac{1}{2}\frac{M}{R}\int_{x=-R}^{R} y^2\mathrm{d}x = \frac{M}{R}\int_{x=0}^{R} y^2\mathrm{d}x = \frac{M}{R}\int_{x=0}^{R} \left(R^2 - x^2\right)\mathrm{d}x$$

$$= \frac{M}{R}\left(R^2\,R - \frac{R^3}{3}\right) \quad \Rightarrow \quad \boxed{I = \frac{2}{3}MR^2}\,. \tag{1.5}$$