1.1

$$L = T - V = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k (x_2 - x_1)^2$$

$$= \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2 - \frac{1}{2} k x_1^2 - \frac{1}{2} k x_2^2 + k x_1 x_2$$

$$= \frac{1}{2} (\dot{x}_1 \ \dot{x}_2) \underbrace{\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}}_{\overleftrightarrow{M}} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} - \frac{1}{2} (x_1 \ x_2) \underbrace{k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}_{\overleftrightarrow{K}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\overleftrightarrow{K} \vec{q} + \overleftrightarrow{M} \ddot{\vec{q}} = 0 \qquad \vec{q} = \vec{a} \cos(\omega t - \delta) \Rightarrow \overleftrightarrow{K} \vec{a} - \omega^2 \overleftrightarrow{M} \vec{a} = 0$$

$$\Rightarrow \begin{pmatrix} k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\Rightarrow \det \begin{vmatrix} k - \omega^2 m & -k \\ -k & k - \omega^2 m \end{vmatrix} = 0 \Rightarrow (k - \omega^2 m)^2 - k^2 = 0$$

$$\Rightarrow (\omega^2 m)^2 - 2k \omega^2 m = 0 \Rightarrow \omega^2 \left(\omega^2 - \frac{2k}{m} \right) = 0$$

$$\Rightarrow \quad \omega_1^2 = 0 \Rightarrow k a_1 - k a_2 = 0 \Rightarrow a_1 = a_2$$

$$\omega_2^2 = \frac{2k}{m} \Rightarrow (k - 2k) a_1 - k a_2 = 0 \Rightarrow a_1 = -a_2$$

$$\Rightarrow \boxed{\vec{Q}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A_1 t + \delta_1) \quad \omega_1 = 0 \qquad \vec{Q}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cos(\omega_2 t - \delta_2) \quad \omega_2^2 = \frac{2k}{m}}$$

1.2 $\vec{q}(t) = \begin{pmatrix}1\\1\end{pmatrix}(A_1 t + \delta_1) + \begin{pmatrix}1\\-1\end{pmatrix} A_2 \cos(\omega_2 t - \delta_2)$

$$\boxed{\begin{aligned} \vec{q}(0) &= \begin{pmatrix}b\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}\delta_1 + \begin{pmatrix}1\\-1\end{pmatrix} A_2 \cos\delta_2 \\ \dot{\vec{q}}(0) &= \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} A_1 + \begin{pmatrix}1\\-1\end{pmatrix} A_2 \sin\delta_2 \end{aligned}}$$

4 equations 4 unknowns

Easyer in Normal Coordinates

Define $\eta_1 \equiv x_1 + x_2$ $\quad \eta_2 \equiv x_1 - x_2$

change variables

$\Rightarrow \eta_1(t) = 2(A_1 t + \delta_1) \qquad \eta_2(t) = 2A_2 \cos(\omega_2 t - \delta_2)$

$\eta_1(0) = b + 0$

$\Rightarrow b = 2\delta_1$

$\Rightarrow \delta_1 = \frac{b}{2}$

$\dot{\eta}_1(0) = 0$

$\Rightarrow 0 = 2A_1$

$\Rightarrow A_1 = 0$

$\left.\begin{aligned} \eta_2(0) &= b - 0 = 2A_2\cos\delta_2 \\ \dot{\eta}_2(0) &= 0 = 2A_2 \sin\delta_2 \end{aligned}\right\} \Rightarrow A_2 = \frac{b}{2},\ \Rightarrow \delta_2 = 0$

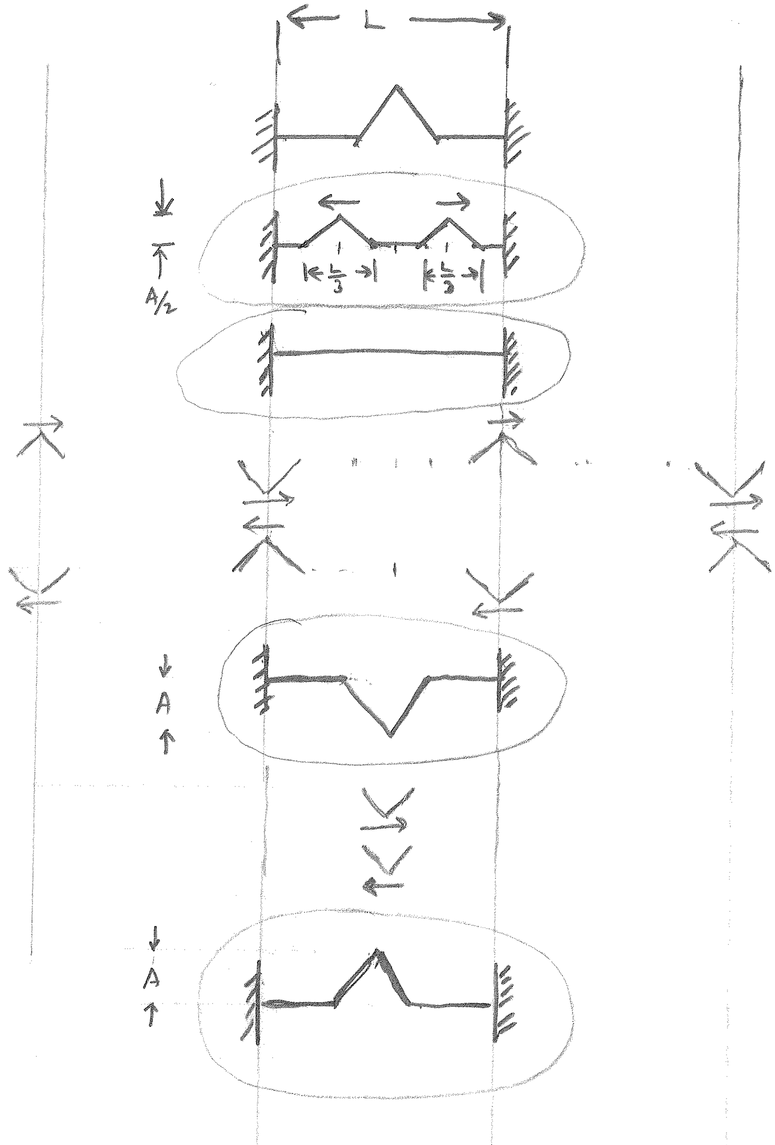
$$\Rightarrow \boxed{\vec{q}(t) = \begin{pmatrix}1\\1\end{pmatrix}\frac{b}{2} + \begin{pmatrix}1\\-1\end{pmatrix}\frac{b}{2}\cos\omega_2 t, \quad \omega_2^2 = \frac{2k}{m}}$$

2.1

$t=0$
$d=0$

$t=\frac{L}{4v}$
$d=\frac{L}{4}$

$t=\frac{L}{2v}$
$d=\frac{L}{2}$



$t=\frac{L}{v}$
$d=L$

$t=\frac{2L}{v}$
$d=2L$
one period

2.2

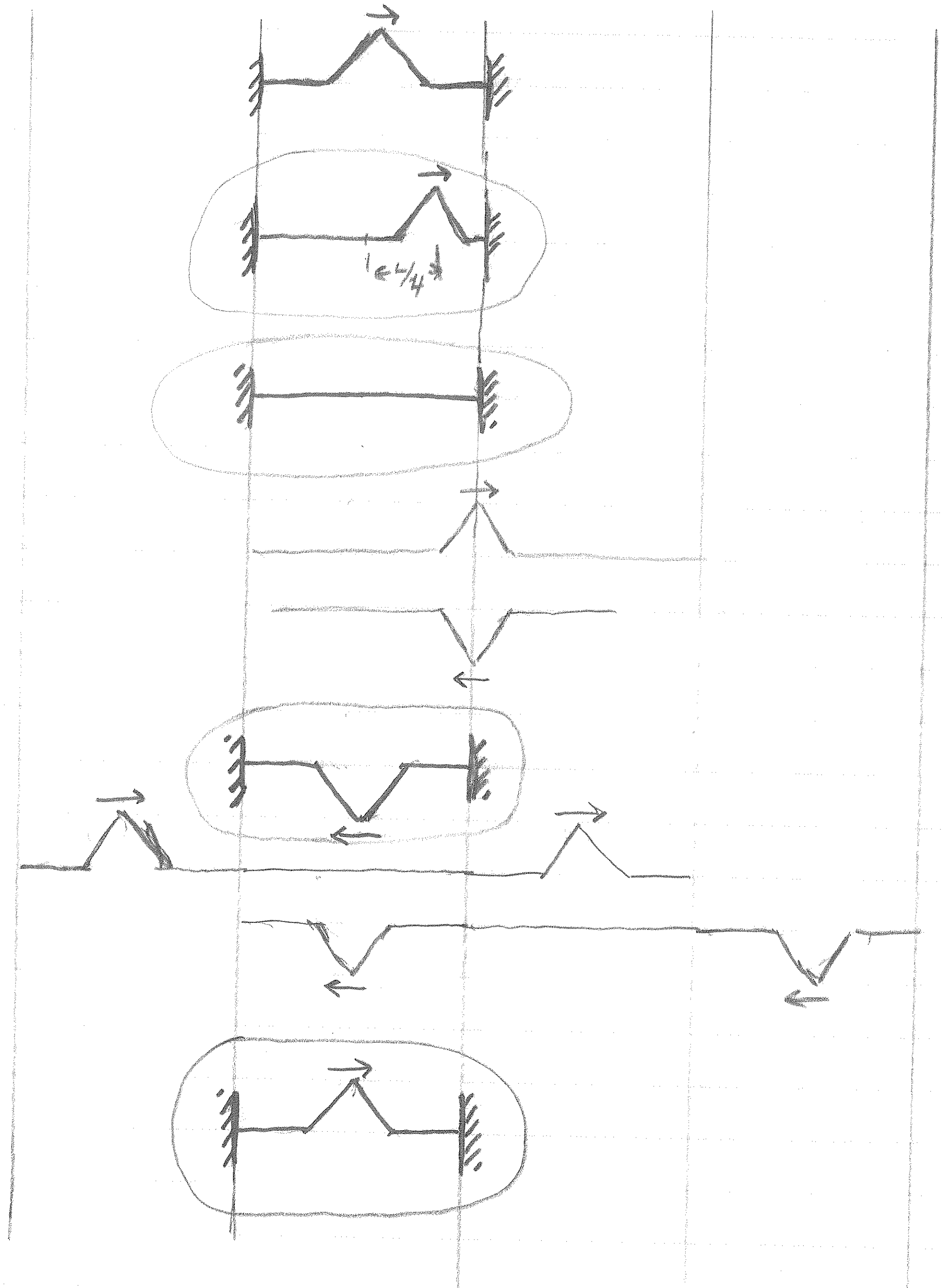
$t=0$
$d=0$

$t=\frac{L}{4v}$
$d=\frac{L}{4}$

$t=\frac{L}{2v}$
$d=\frac{L}{2}$

$t=\frac{L}{v}$
$d=L$

$t=\frac{2L}{v}$
$d=2L$
one period



3.1

$$T = \frac{1}{2} m \dot{q}_1^2 + \frac{1}{2} m \dot{q}_2^2 + \cdots$$

$$V = \frac{1}{2} k q_1^2 + \frac{1}{2} k (q_2 - q_1)^2 + \cdots$$

$$+ \frac{1}{2} m g l \left(\frac{q_1}{l}\right)^2 + \frac{1}{2} m g l \left(\frac{q_2}{l}\right)^2 + \cdots$$

$$\Rightarrow m\ddot{q}_k = -k(q_k - q_{k-1}) + k(q_{k+1} - q_k) - \frac{mg}{l} q_k$$

$$q_k = a_k \cos\omega t \quad \Rightarrow \ddot{q}_k = -\omega^2 a_k \cos\omega t$$

$$\Rightarrow -m\omega^2 a_k = -k(2a_k - a_{k+1} - a_{k-1}) - \frac{mg}{l} a_k$$

$$\Rightarrow -\omega^2 a_k = -\frac{k}{m}(2a_k - a_{k+1} - a_{k-1}) - \frac{g}{l} a_k$$

$$a_k = A \sin k\phi$$

$$\Rightarrow -\omega^2 \sin k\phi = -\left(\frac{2k}{m} + \frac{g}{l}\right) \sin k\phi + \frac{k}{m} \sin[(k+1)\phi] + \frac{k}{m} \sin[(k-1)\phi]$$

$$\Rightarrow \omega^2 \sin k\phi = \left(\frac{2k}{m} + \frac{g}{l}\right) \sin k\phi - \frac{k}{m}\left\{\sin[(k+1)\phi] + \sin[(k-1)\phi]\right\}$$

$$\Rightarrow \omega^2 \cancel{\sin k\phi} = \left(\frac{2k}{m} + \frac{g}{l}\right) \cancel{\sin k\phi} - \frac{2k}{m} \cancel{\sin k\phi} \cos\phi$$

$$\Rightarrow \omega^2 = \frac{g}{l} + \frac{2k}{m}(1 - \cos\phi) = \frac{g}{l} + \frac{4k}{m} \sin^2\frac{\phi}{2}$$

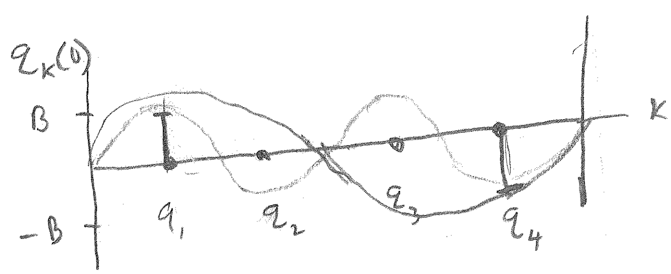
$$a_{n+1} = 0 \quad \Rightarrow \quad 0 = A \sin[(n+1)\phi] \Rightarrow \phi = \frac{N\pi}{n+1} \quad \Rightarrow a_k = A \sin\left(\frac{kN\pi}{n+1}\right)$$

$$\boxed{\omega_N^2 = \frac{g}{l} + \frac{4k}{m}\sin^2\left(\frac{N\pi}{2n+2}\right) \qquad \vec{a}_N = \begin{pmatrix} a_{1N} \\ a_{2N} \\ \vdots \\ a_{nN} \end{pmatrix} \qquad a_{kN} = \sin\left(\frac{kN\pi}{n+1}\right)}$$

3.2 $$\vec{q}(t) = \sum_N A_N \vec{a}_N \cos(\omega_N t - \delta_N)$$

$$\vec{q}(0) = \begin{pmatrix} B \\ 0 \\ 0 \\ -B \end{pmatrix} \qquad \dot{\vec{q}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Setting $\delta_N = 0$ for $N = 1, 2, 3, 4$ $\Rightarrow \dot{\vec{q}}(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$



The initial conditions are anti symmetric about the center therefore there are no symmetric modes $\Rightarrow A_1 = A_3 = 0$

$$\Rightarrow \vec{q}(0) = \begin{pmatrix} B \\ 0 \\ 0 \\ -B \end{pmatrix} = A_2 \begin{pmatrix} \sin\frac{2\pi}{5} \\ \sin\frac{4\pi}{5} \\ \sin\frac{6\pi}{5} \\ \sin\frac{8\pi}{5} \end{pmatrix} + A_4 \begin{pmatrix} \sin\frac{4\pi}{5} \\ \sin\frac{8\pi}{5} \\ \sin\frac{12\pi}{5} \\ \sin\frac{16\pi}{5} \end{pmatrix}$$

Now fix the $\frac{kN\pi}{n+1}$ phase numbers so there are simular

$$\Rightarrow \quad \vec{q}(0) = \begin{pmatrix} B \\ 0 \\ 0 \\ -B \end{pmatrix} = A_2 \begin{pmatrix} \sin\frac{2\pi}{5} \\ \sin\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \end{pmatrix} + A_4 \begin{pmatrix} \sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \\ -\sin\frac{4\pi}{5} \end{pmatrix}$$

Same Same

Shows two of the equations are extra

$$\Rightarrow \quad A_2 \sin\frac{2\pi}{5} + A_4 \sin\frac{4\pi}{5} = B$$

$$A_2 \sin\frac{4\pi}{5} = A_4 \sin\frac{2\pi}{5}$$

$$\Rightarrow \quad A_2 \sin\frac{2\pi}{5} + \left(A_2 \frac{\sin\frac{4\pi}{5}}{\sin\frac{2\pi}{5}} \right) \sin\frac{4\pi}{5} = B$$

$$\Rightarrow A_2 = B \left[\sin\frac{2\pi}{5} + \frac{\sin^2\frac{4\pi}{5}}{\sin\frac{2\pi}{5}} \right]^{-1}$$

$$A_4 = \frac{\sin\frac{4\pi}{5}}{\sin\frac{2\pi}{5}} \left[\sin\frac{2\pi}{5} + \frac{\sin^2\frac{4\pi}{5}}{\sin\frac{2\pi}{5}} \right]^{-1} B$$

$$\Rightarrow \quad A_4 = \left[\frac{\sin^2\frac{2\pi}{3}}{\sin\frac{4\pi}{5}} + \sin\frac{4\pi}{5} \right]^{-1} B$$

$$\vec{q}(t) = \frac{B}{\sin\frac{2\pi}{5} + \frac{\sin^2\frac{4\pi}{5}}{\sin\frac{2\pi}{5}}} \begin{pmatrix} \sin\frac{2\pi}{5} \\ \sin\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \end{pmatrix} \cos(\omega_2 t)$$

$$+ \frac{B}{\sin\frac{4\pi}{5} + \frac{\sin^2\frac{2\pi}{3}}{\sin\frac{4\pi}{5}}} \begin{pmatrix} \sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \\ -\sin\frac{4\pi}{5} \end{pmatrix} \cos(\omega_4 t)$$

$$\omega_2^2 = \frac{g}{l} + \frac{4k}{m} \sin^2\frac{\pi}{5}, \quad \omega_4 = \frac{g}{l} + \frac{4K}{m} \sin^2\frac{2\pi}{5}$$

$$\sin\frac{2\pi}{5} = \frac{\sqrt{2(5+\sqrt{5})}}{4} \qquad \sin\frac{4\pi}{5} = \frac{\sqrt{2(5-\sqrt{5})}}{4}$$

$$A_2 = \frac{\sin\frac{2\pi}{5}\, B}{\sin^2\frac{2\pi}{5} + \sin^2\frac{4\pi}{5}} = \frac{\sqrt{2(5+\sqrt{5})}}{4} B \left(\frac{2(5+\sqrt{5})}{16} + \frac{2(5-\sqrt{5})}{16} \right)^{-1}$$

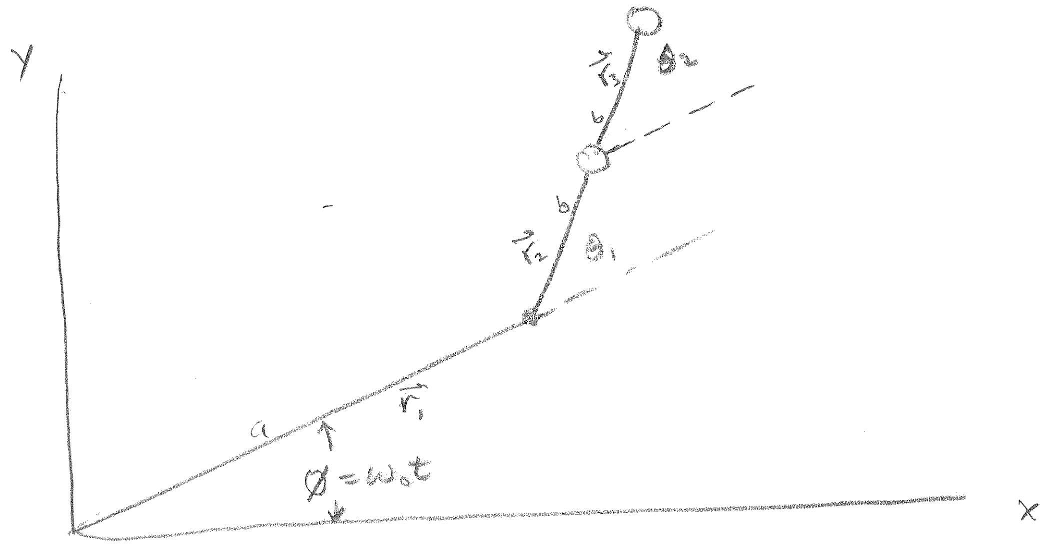
$$= \frac{\sqrt{2(5+\sqrt{5})}}{\cancel{4}} B \frac{\overset{4}{\cancel{16}}}{10+10} = \frac{\sqrt{2(5+\sqrt{5})}}{5} B$$

$$A_4 = A_2 \frac{\sin\frac{4\pi}{5}}{\sin\frac{2\pi}{5}} = \cancel{\frac{\sqrt{2(5+\sqrt{5})}}{5}} B \frac{\sqrt{2(5-\sqrt{5})}}{\cancel{\sqrt{2(5+\sqrt{5})}}}$$

$$= \frac{\sqrt{2(5-\sqrt{5})}}{5} B$$

$$\Rightarrow \boxed{\vec{q}(t) = \frac{\sqrt{2(5+\sqrt{5})}}{5} B \begin{pmatrix} \sin\frac{2\pi}{5} \\ \sin\frac{4\pi}{5} \\ -\sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \end{pmatrix} \cos(\omega_2 t) + \frac{\sqrt{2(5-\sqrt{5})}}{5} B \begin{pmatrix} \sin\frac{4\pi}{5} \\ -\sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} \\ -\sin\frac{4\pi}{5} \end{pmatrix} \cos(\omega_4 t)}$$

4.1 Lagrangian



$$T = T_1 + T_2 = \tfrac{1}{2} m v_1^2 + \tfrac{1}{2} m v_2^2$$

$$v_1^2 = (\dot{\vec{r}}_1 + \dot{\vec{r}}_2)\cdot(\dot{\vec{r}}_1 + \dot{\vec{r}}_2)$$

$$\vec{r}_1 + \vec{r}_2 = (a\cos\phi + b\cos(\phi+\theta_1))\,\hat{x} + (a\sin\phi + b\sin(\phi+\theta_1))\,\hat{y}$$

$$\dot{\vec{r}}_1 + \dot{\vec{r}}_2 = (-a\dot{\phi}\sin\phi - b(\dot{\phi}+\dot{\theta}_1)\sin(\phi+\theta_1))\,\hat{x}$$
$$+ (a\dot{\phi}\cos\phi + b(\dot{\phi}+\dot{\theta}_1)\cos(\phi+\theta_1))\,\hat{y}$$

$$v_1^2 = a^2\dot{\phi}^2\sin^2\phi + b^2(\dot{\phi}+\dot{\theta}_1)^2\sin^2(\phi+\theta_1)$$
$$+ 2ab\,\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\sin\phi\sin(\phi+\theta_1)$$
$$+ a^2\dot{\phi}^2\cos^2\phi + b^2(\dot{\phi}+\dot{\theta}_1)^2\cos^2(\phi+\theta_1)$$
$$+ 2ab\,\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\cos\phi\cos(\phi+\theta_1)$$

$\Downarrow$

$$v_1^2 = a^2\dot{\phi}^2 + b^2(\dot{\phi}+\dot{\theta}_1)^2 + 2ab\,\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\cos\theta_1$$

$$v_2^2 = (\dot{\vec{r}}_1 + \dot{\vec{r}}_2 + \dot{\vec{r}}_3)\cdot(\dot{\vec{r}}_1 + \dot{\vec{r}}_2 + \dot{\vec{r}}_3)$$

$$\vec{r}_1 + \vec{r}_2 + \vec{r}_3 = (a\cos\phi + b\cos(\phi+\theta_1) + b\cos(\phi+\theta_2))\,\hat{x}$$
$$+ (a\sin\phi + b\sin(\phi+\theta_1) + b\sin(\phi+\theta_2))\,\hat{y}$$

$$\dot{\vec{r}}_1 + \dot{\vec{r}}_2 + \dot{\vec{r}}_3 = -\left(a\dot{\phi}\sin\phi + b(\dot{\phi}+\dot{\theta}_1)\sin(\phi+\theta_1) + b(\dot{\phi}+\dot{\theta}_2)\sin(\phi+\theta_2)\right)\hat{x}$$

$$+ \left(a\dot{\phi}\cos\phi + b(\dot{\phi}+\dot{\theta}_1)\cos(\phi+\theta_1) + b(\dot{\phi}+\dot{\theta}_2)\cos(\phi+\theta_2)\right)\hat{y}$$

$$v_2^2 = (\dot{\vec{r}}_1 + \dot{\vec{r}}_2 + \dot{\vec{r}}_3)\cdot(\dot{\vec{r}}_1 + \dot{\vec{r}}_2 + \dot{\vec{r}}_3)$$

$$v_2^2 = a^2\dot{\phi}^2\sin^2\phi + b^2(\dot{\phi}+\dot{\theta}_1)^2\sin^2(\phi+\theta_1) + b^2(\dot{\phi}+\dot{\theta}_2)^2\sin^2(\phi+\theta_2)$$

$$+ 2ab\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\sin\phi\sin(\phi+\theta_1) + 2ab\dot{\phi}(\dot{\phi}+\dot{\theta}_2)\sin\phi\sin(\phi+\theta_2)$$

$$+ 2b^2(\dot{\phi}+\dot{\theta}_1)(\dot{\phi}+\dot{\theta}_2)\sin(\phi+\theta_1)\sin(\phi+\theta_2)$$

$$+ a^2\dot{\phi}^2\cos^2\phi + b^2(\dot{\phi}+\dot{\theta}_1)^2\cos^2(\phi+\theta_1) + b^2(\dot{\phi}+\dot{\theta}_2)^2\cos^2(\phi+\theta_2)$$

$$+ 2ab\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\cos\phi\cos(\phi+\theta_1) + 2ab\dot{\phi}(\dot{\phi}+\theta_2)\sin\phi\sin(\phi+\theta_2)$$

$$+ 2b^2(\dot{\phi}+\dot{\theta}_1)(\phi+\theta_2)\cos(\phi+\theta_1)\cos(\phi+\theta_2)$$

$$v_2^2 = a^2\dot{\phi}^2 + b^2(\dot{\phi}+\dot{\theta}_1)^2 + b^2(\dot{\phi}+\dot{\theta}_2)^2$$

$$+ 2ab\dot{\phi}(\dot{\phi}+\dot{\theta}_1)\cos\theta_1 + 2ab\dot{\phi}(\dot{\phi}+\dot{\theta}_2)\cos\theta_2$$

$$+ 2b^2(\dot{\phi}+\dot{\theta}_1)(\dot{\phi}+\dot{\theta}_2)\cos(\theta_2-\theta_1)$$

$$\Rightarrow T = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 \qquad \dot{\phi} = \omega_0 \qquad V = 0$$

$$\boxed{\begin{aligned} L = {} & ma^2\omega_0^2 + mb^2(\omega_0+\dot{\theta}_1)^2 + 2mab\,\omega_0(\omega_0+\dot{\theta}_1)\cos\theta_1 \\ & + \tfrac{1}{2}mb^2(\omega_0+\dot{\theta}_2)^2 + mab\,\omega_0(\omega_0+\dot{\theta}_2)\cos\theta_2 \\ & + mb^2(\omega_0+\dot{\theta}_1)(\omega_0+\dot{\theta}_2)\cos(\theta_2-\theta_1) \end{aligned}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial L}{\partial \theta_1} \Rightarrow \frac{d}{dt}\Big[2mb^2(\omega_0+\dot\theta_1) + 2mab\omega_0\cos\theta_1 + mb^2(\omega_0+\dot\theta_2)\cos(\theta_2-\theta_1)\Big]$$

$$= -2mab\omega_0(\omega_0+\dot\theta_1)\sin\theta_1 + mb^2(\omega_0+\dot\theta_1)(\omega_0+\dot\theta_2)\sin(\theta_2-\theta_1)$$

$$\Rightarrow 2mb^2\ddot\theta_1 - 2mab\omega_0\cancelto{0}{\dot\theta_1\sin\theta_1} + mb^2\ddot\theta_2\cancelto{1}{\cos(\theta_2-\theta_1)}$$

$$- mb^2(\omega_0+\dot\theta_2)\cancelto{0}{(\dot\theta_2-\dot\theta_1)\sin(\theta_2-\theta_1)}$$

$$= -2mab\omega_0^2\sin\theta_1 - \cancelto{0}{2mab\omega_0\dot\theta_1\sin\theta_1}$$

$$+ mb^2\omega_0^2\sin(\theta_2-\theta_1) + \cancelto{0}{mb^2\omega_0\dot\theta_1\sin(\theta_2-\theta_1)}$$

$$+ mb^2\dot\theta_1\cancelto{0}{\dot\theta_2\sin(\theta_2-\theta_1)}$$

getting rid of small terms and expanding

$$\Rightarrow 2mb^2\ddot\theta_1 + mb^2\ddot\theta_2 = -(2a+b)mb\omega_0^2\theta_1 + mb^2\omega_0^2\theta_2$$

$$\Rightarrow \boxed{2b\ddot\theta_1 + b\ddot\theta_2 = -(2a+b)\omega_0^2\theta_1 + b\omega_0^2\theta_2}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = \frac{\partial L}{\partial \theta_1} \Rightarrow \frac{d}{dt}\Big[mb^2(\omega_0+\dot{\theta}_2) + mab\omega_0\cos\theta_2$$

$$+ mb^2(\omega_0+\dot{\theta}_1)\cos(\theta_2-\theta_1)\Big]$$

$$= -mab\omega_0(\omega_0+\dot{\theta}_2)\sin\theta_2 - mb^2(\omega_0+\dot{\theta}_1)(\omega_0+\dot{\theta}_2)\sin(\theta_2-\theta_1)$$

$$\Rightarrow mb^2\ddot{\theta}_2 - mab\omega_0\cancelto{0}{\dot{\theta}_2\sin\theta_2} + mb^2\ddot{\theta}_1\cancelto{1}{\cos(\theta_2-\theta_1)}$$

$$- mb^2(\omega_0+\dot{\theta}_1)\cancelto{0}{(\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)}$$

$$= -mab\omega_0^2\sin\theta_2 - mab\omega_0\cancelto{0}{\dot{\theta}_2\sin\theta_2} - mb^2\omega_0^2\sin(\theta_2-\theta_1)$$

$$- mb^2\omega_0\cancelto{0}{\dot{\theta}_1\sin(\theta_2-\theta_1)} - mb^2\omega_0\cancelto{0}{\dot{\theta}_2\sin(\theta_2-\theta_1)} - mb^2\cancelto{0}{\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)}$$

getting rid of small terms and expanding

$$\Rightarrow mb^2\ddot{\theta}_2 + mb^2\ddot{\theta}_1 = -mab\omega_0^2\theta_2 - mb^2\omega_0^2\theta_2 + mb^2\omega_0^2\theta_1$$

$$\Rightarrow \boxed{b\ddot{\theta}_2 + b\ddot{\theta}_1 = -(a+b)\omega_0^2\theta_2 + b\omega_0^2\theta_1}$$

$$\Rightarrow \boxed{\omega_0^2\begin{pmatrix} 2a+b & -b \\ -b & a+b \end{pmatrix}\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} + b\begin{pmatrix} 2 & 1 \\ 1 & 1\end{pmatrix}\begin{pmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_2\end{pmatrix} = 0}$$

4.3 $\boxed{\overleftrightarrow{M} = b\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \qquad \overleftrightarrow{K} = \omega_0^2 \begin{pmatrix} 2a+b & -b \\ -b & a+b \end{pmatrix}}$ 13

4.4 $\vec{q} = \vec{a}\cos\omega t \qquad \ddot{\vec{q}} = -\omega^2 \vec{a}\cos\omega t$

$$\overleftrightarrow{K}\vec{q} + \overleftrightarrow{M}\ddot{\vec{q}} = 0 \Rightarrow (\overleftrightarrow{K} - \omega^2\overleftrightarrow{M})\vec{a} = 0$$

$$\Rightarrow \begin{pmatrix} \omega_0^2(2a+b) - \omega^2 2b & -\omega_0^2 b - \omega^2 b \\ -\omega_0^2 b - \omega^2 b & \omega_0^2(a+b) - \omega^2 b \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\det|\overleftrightarrow{K} - \omega^2\overleftrightarrow{M}| = 0 \Rightarrow [\omega_0^2(2a+b) - \omega^2 2b][\omega_0^2(a+b) - \omega^2 b] - (\omega_0^2 + \omega^2)^2 b^2 = 0$$

$$\Rightarrow \cancel{2}b^2(\omega^2)^2 - [2b(a+b) + b(2a+b)]\omega_0^2\omega^2 + (\omega_0^2)^2(2a+b)(a+b) - (\omega_0^2)^2 b^2 - 2b^2\omega_0^2\omega^2 - \cancel{b^2(\omega^2)^2} = 0$$

$$\Rightarrow b^2(\omega^2)^2 - [4ab + 5b^2]\omega_0^2\omega^2 + (2a^2 + 3ab)(\omega_0^2)^2 = 0$$

$$\Rightarrow \omega^2 = \frac{(4ab + 5b^2)\omega_0^2 \pm \sqrt{16a^2b^2 + 40ab^3 + 25b^4 - 4b^2(2a^2+3ab)}\,\omega_0^2}{2b^2}$$

$$\Rightarrow \frac{\omega^2}{\omega_0^2} = 2\frac{a}{b} + \frac{5}{2} \pm \frac{\sqrt{16a^2b^2 + 40ab^3 + 25b^4 - 8a^2b^2 - 12ab^3}\,\omega_0^2}{2b^2}$$

$$\frac{\omega^2}{\omega_0^2} = \frac{5}{2} + 2\frac{a}{b} \pm \frac{1}{2}\sqrt{8\left(\frac{a}{b}\right)^2 + 28\frac{a}{b} + 25}$$

$$\Rightarrow \boxed{\omega_\pm^2 = \omega_0^2\left[\frac{5}{2} + 2\frac{a}{b} \pm \frac{1}{2}\underbrace{\sqrt{25 + 28\frac{a}{b} + 8\left(\frac{a}{b}\right)^2}}_{\frac{\sqrt{\ }}{b}}\right]}$$

for $(\overleftrightarrow{K} - \omega^2\overleftrightarrow{M})\vec{a} = 0$

top equation

$$\Rightarrow \left[\omega_0^2(2a+b) - \omega^2 2b\right] a_1 = (\omega_0^2 + \omega^2) b\, a_2$$

$$\Rightarrow \left(2a + b - 5b - 4a \mp \sqrt{\ }\right) a_1 = b + \frac{1}{2}\left(5b + 4a \pm \sqrt{\ }\right) a_2$$

$$\Rightarrow \left(-2a - 4b \mp \sqrt{\ }\right) a_1 = \frac{1}{2}\left(7b + 4a \pm \sqrt{\ }\right) a_2$$

$$\Rightarrow \frac{a_1}{a_2} = -\frac{7b + 4a \pm \sqrt{\ }}{4a + 8b \pm 2\sqrt{\ }}$$

$$\Rightarrow \boxed{\vec{a}_\pm = \begin{pmatrix} -\dfrac{7b + 4a \pm \sqrt{25b^2 + 28ab + 8a^2}}{4a + 8b \pm 2\sqrt{25b^2 + 28ab + 8a^2}} \\ 1 \end{pmatrix}}$$