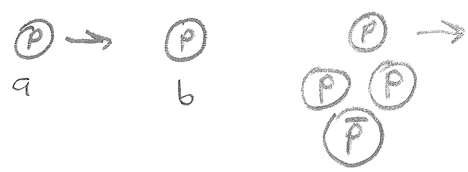
Exam II Phys 3356

1.

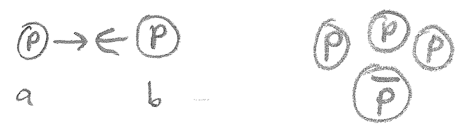
In Lab frame

Before After



In center of Momentum frame (CM)

Before After



The 4-momentum scalar product is invariant and constant

$$P_{cm}^2 = P_{Lab}^2$$

$$\underset{\text{after}}{P_{cm}} = \left(0, i\frac{E_{cm}}{c}\right) \qquad \underset{\text{Before}}{P_{Lab}} = P_{a_{Lab}} + P_{b_{Lab}}$$

$$= \left(\vec{P}, i\frac{E_a}{c}\right) + \left(0, i m_p c\right)$$

$$\underset{\text{after}}{P_{cm}^2} = \underset{\text{Before}}{P_{Lab}^2} \Rightarrow -\frac{E_{cm(after)}^2}{c^2} = \left(P_{a_{Lab}} + P_{b_{Lab}}\right)^2$$

$$-\frac{(4m_p c^2)^2}{c^2} = P_{a_{Lab}}^2 + P_{b_{Lab}}^2 + 2 P_{a_{Lab}} \cdot P_{b_{Lab}}$$

$$-16 m_p^2 c^2 = \overbrace{-m_p^2 c^2}^{} + \overbrace{(-m_p^2 c^2)}^{} - 2\left(\frac{E_a}{c}\right)(m_p c)$$

$$\Rightarrow \boxed{E_a = 7 m_p c^2} \qquad \boxed{T = E_a - m_p c^2 = 6 m_p c^2}$$

1. by Thorton was 14-24

The minimum energy will occur when the four particles are all at rest in the center of the mass system after the collision.

Conservation of energy gives (in the CM system)

$$2E_p = 4m_pc^2$$

or

$$E_{p,\text{CM}} = 2m_pc^2 = 2E_0$$

which implies $\gamma = 2$ or $\beta = \sqrt{3}/2$

To find the energy required in the lab system (one proton at rest initially), we transform back to the lab

$$E = \gamma\left(E' + vp_1'\right) \tag{1}$$

The velocity of K'(CM) with respect to K(lab) is just the velocity of the proton in the K' system. So $u = v$.

Then

$$vp_1' = v\left(p_{\text{CM}}\right) = v\left(\gamma mu\right) = \gamma mv^2 = \gamma mc^2\beta^2$$

Since $\gamma = 2,\ \beta = \sqrt{3}/2$,

$$vp_1' = \frac{3}{2}E_0$$

Substituting into (1)

$$E_{\text{lab}} = \gamma\left(2E_0 + \frac{3}{2}E_0\right) = 2\left[\frac{7}{2}E_0\right] = 7E_0$$

The minimum proton energy in the lab system is 7 m_pc^2 , of which 6 m_pc^2 is kinetic energy.

2.1

$$\dot{x} = y \qquad\qquad = f(x, y)$$

$$\dot{y} = -\sin x - y = g(x, y)$$

$$f(x_0, y_0) = 0 \quad \Rightarrow \quad y_0 = 0$$

$$g(x_0, y_0) = 0 \quad \Rightarrow \quad \sin x_0 = 0$$

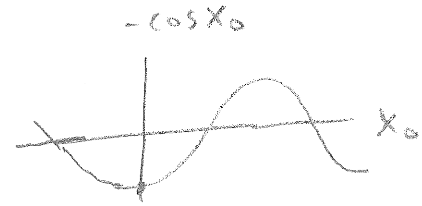
$$\Rightarrow \boxed{(x_0, y_0) = (n\pi, 0)}$$

$$n = \ldots -2, -1, 0, 1, 2 \ldots$$

2.2

$$\begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} \frac{\partial f(x_0, y_0)}{\partial x} & \frac{\partial f(x_0, y_0)}{\partial y} \\ \frac{\partial g(x_0, y_0)}{\partial x} & \frac{\partial g(x_0, y_0)}{\partial y} \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\cos x_0 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}$$



$$\Rightarrow \boxed{\begin{pmatrix} \dot{\mu} \\ \dot{\nu} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mp 1 & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix}}$$

2.3

$$\begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t} \quad \Rightarrow \quad \lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \mp 1 & -1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\Rightarrow \quad 0 = \begin{pmatrix} -\lambda & 1 \\ \mp 1 & -1-\lambda \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad \Rightarrow \quad \begin{vmatrix} -\lambda & 1 \\ \mp 1 & -1-\lambda \end{vmatrix} = 0$$

2.3 (continued)

$$\Rightarrow \lambda(\lambda+1) \pm 1 = 0 \Rightarrow \lambda^2 + \lambda \pm 1 = 0$$

$$\Rightarrow \lambda = \frac{-1 \pm \sqrt{1 - 4(\pm 1)}}{2} = \underbrace{-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}}_{\substack{\text{UPPER sign} \\ n = \text{even}}}, \underbrace{\frac{\pm\sqrt{5} - 1}{2}}_{\substack{\text{Lower sign} \\ n = \text{odd}}}$$

from Before $\lambda a_1 = a_2$

$a_2 = \lambda a_1$

For $\begin{pmatrix} -\lambda & 1 \\ +1 & -1-\lambda \end{pmatrix}$ matrix or $n = \text{odd}$

$$\begin{pmatrix} \mu \\ \gamma \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{pmatrix} e^{\left(\frac{\sqrt{5}-1}{2}\right)t} + A_2 \begin{pmatrix} 1 \\ -\frac{(\sqrt{5}+1)}{2} \end{pmatrix} e^{-\left(\frac{\sqrt{5}+1}{2}\right)t}$$

For $\begin{pmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix}$ matrix or $n = \text{even}$

$$\boxed{\begin{pmatrix} \mu \\ \gamma \end{pmatrix} = A_1 \begin{pmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} e^{-\frac{1}{2}t} e^{+i\frac{\sqrt{3}}{2}t} + A_2 \begin{pmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} e^{-\frac{1}{2}t} e^{-i\frac{\sqrt{3}}{2}t}}$$

A_1 and A_2 must is complex so that $\mu(t)$ and $\gamma(t)$ are real. We can write the answer in terms of real numbers with $A_1 = \overline{A_2}$.

2.3 Appendex

Showing That $A_1 = \overline{A_2}$

(Some people asked for this)

(where bar is complex conjugate)

for $A_1 f(t) + A_2 \overline{f}(t)$ to be real for all t.

$\overline{f}$ = complex conjugate of f.

Let $A_1 = B_1 + i C_1$, $A_2 = B_2 + i C_2$

and $f = f_1 + i f_2$ where

$B_1\ C_1\ B_2\ C_2\ f_1\ f_2$ are real.

$$A_1 f + A_2 \overline{f} = (B_1 + i C_1)(f_1 + i f_2) + (B_2 + i C_2)(f_1 + i f_2)$$

$$= B_1 f_1 - C_1 f_2 + B_2 f_1 + C_2 f_2 + i[B_1 f_2 + C_1 f_1 - B_2 f_2 + C_2 f_1]$$

for this to be real $B_1 f_2 + C_1 f_1 - B_2 f_2 + C_2 f_1 = 0$

$$\Rightarrow f_2(B_1 - B_2) + f_1(C_1 + C_2) = 0$$

For this to be true at all t $B_1 - B_2 = 0$, $C_1 + C_2 = 0$

$$\Rightarrow B_1 = B_2 \equiv B \qquad C_1 = -C_2 \equiv C$$

$$\Rightarrow A_1 = B + iC \qquad A_2 = B - iC = \overline{A_1}$$

QED

2.3 (continued)

$$\begin{pmatrix} u \\ v \end{pmatrix} = (B + iC)\begin{pmatrix} 1 \\ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \end{pmatrix} e^{-\frac{1}{2}t} e^{i\frac{\sqrt{3}}{2}t} \leftarrow$$

$$+ \left[(B - iC)\begin{pmatrix} 1 \\ -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{pmatrix} e^{-\frac{1}{2}t} e^{-i\frac{\sqrt{3}}{2}t} \right] \leftarrow$$

complex conjugate

$A = B + iC \qquad \bar{A} = B - iC$ So $\begin{matrix} u(t) \\ v(t) \end{matrix}$ are real now!

B, C are real

$$u = e^{-\frac{1}{2}t}\left[B\, 2\left(e^{i\frac{\sqrt{3}}{2}t} + e^{-i\frac{\sqrt{3}}{2}t}\right) + C\, 2i\left(e^{i\frac{\sqrt{3}}{2}t} - e^{i\frac{\sqrt{3}}{2}t}\right)\right]$$

$$= e^{-\frac{1}{2}t}\left[4B\cos\left(\frac{\sqrt{3}}{2}t\right) - 4C\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

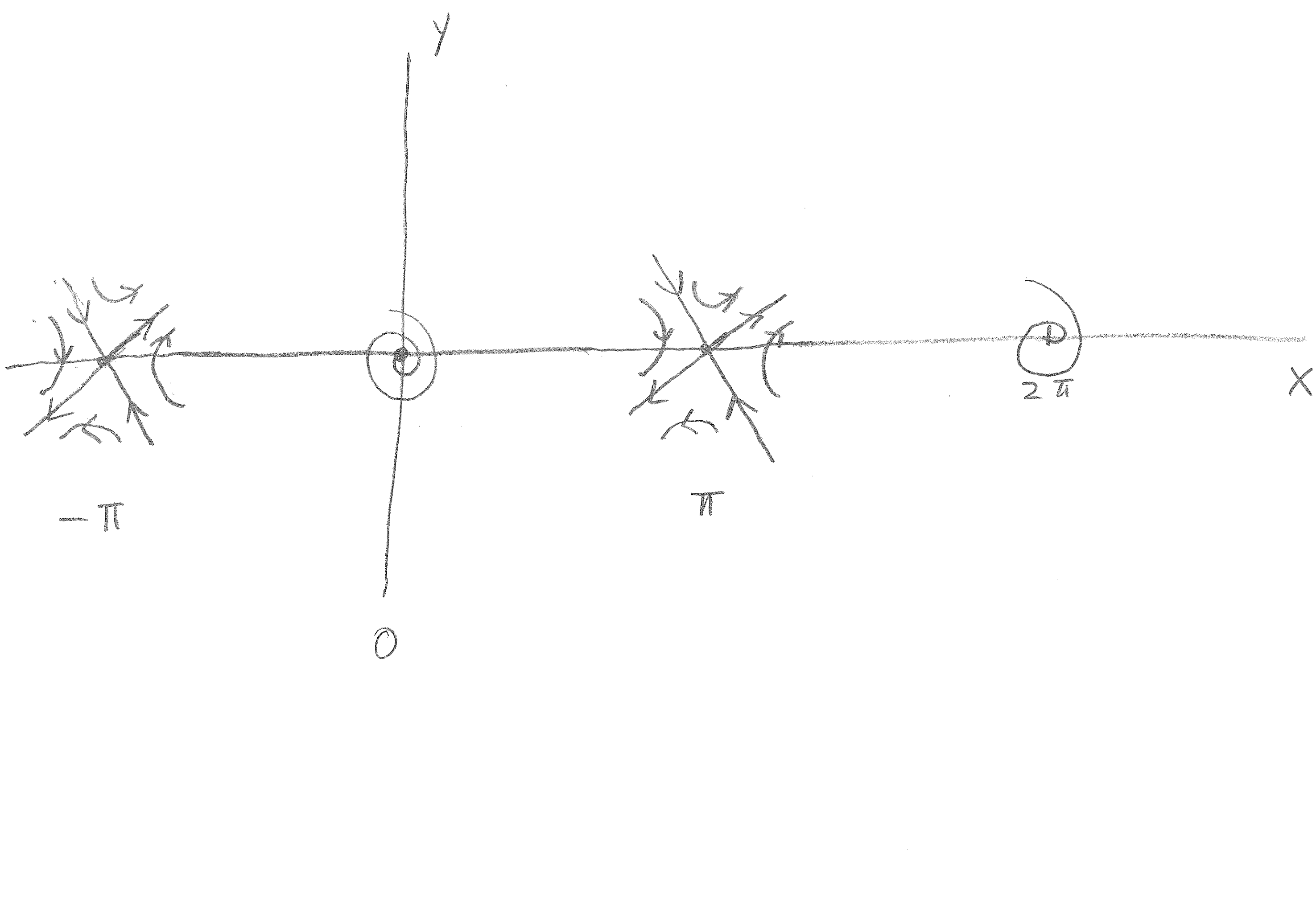
$$\begin{aligned} v = e^{-\frac{1}{2}t}\Big[& B\left(-\frac{1}{\cancel{2}}\right)\cancel{2}\left(e^{i\frac{\sqrt{3}}{2}t} + e^{-i\frac{\sqrt{3}}{2}t}\right) \\ & + B\left(\frac{\sqrt{3}}{\cancel{2}}\right)\cancel{2}i\left(e^{i\frac{\sqrt{3}}{2}t} - e^{-i\frac{\sqrt{3}}{2}t}\right) \\ & + C\left(-\frac{1}{\cancel{2}}\right)\cancel{2}i\left(e^{i\frac{\sqrt{3}}{2}t} - e^{-i\frac{\sqrt{3}}{2}t}\right) \\ & - C\left(\frac{\sqrt{3}}{\cancel{2}}\right)\cancel{2}\left(e^{i\frac{\sqrt{3}}{2}t} + e^{-i\frac{\sqrt{3}}{2}t}\right)\Big] \end{aligned}$$

2.3 (continued)

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{-\frac{1}{2}t} \begin{pmatrix} 4B\cos\left(\frac{\sqrt{3}}{2}t\right) - 4C\sin\left(\frac{\sqrt{3}}{2}t\right) \\ (-2B - 2\sqrt{3}C)\cos\left(\frac{\sqrt{3}}{2}t\right) + (-2\sqrt{3}B + 2C)\sin\left(\frac{\sqrt{3}}{2}t\right) \end{pmatrix}$$

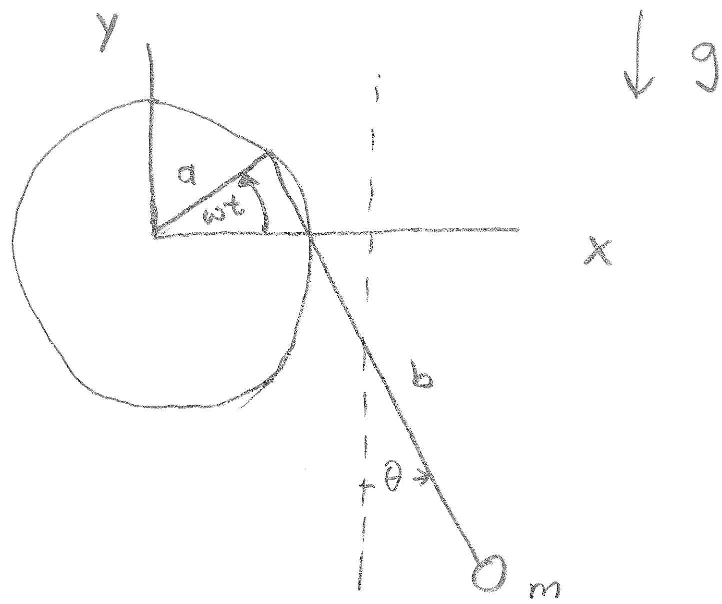
B, C are real divid by 2 would be nice

2.4



Lance Arsenault

3.



3.1

$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 \qquad V = mgy$

$x = a\cos\omega t + b\sin\theta$

$y = a\sin\omega t - b\cos\theta$

$\dot{x} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$

$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta$

$\dot{x}^2 = a^2\omega^2\sin^2\omega t + b^2\dot{\theta}^2\cos^2\theta - 2ab\omega\dot{\theta}\sin\omega t\cos\theta$

$\dot{y}^2 = a^2\omega^2\cos^2\omega t + b^2\dot{\theta}^2\sin^2\theta + 2ab\omega\dot{\theta}\cos\omega t\sin\theta$

$\dot{x}^2 + \dot{y}^2 = a^2\omega^2 + b^2\dot{\theta}^2 + 2ab\omega\dot{\theta}\sin(\theta - \omega t)$

$$\boxed{\begin{aligned} L = {} & \tfrac{1}{2} m a^2\omega^2 + \tfrac{1}{2} m b^2\dot{\theta}^2 + mab\omega\dot{\theta}\sin(\theta-\omega t) \\ & - mga\sin\omega t + mgb\cos\theta \end{aligned}}$$

3.2 $$\boxed{p_\theta \equiv \frac{\partial L}{\partial \dot\theta} = mb^2\dot\theta + mab\omega\sin(\theta-\omega t)}$$

3.3
$$H = \dot\theta p_\theta - L = \dot\theta\, mb^2\dot\theta + mab\omega\dot\theta\sin(\theta-\omega t)$$
$$-\frac{1}{2}ma^2\omega^2 - \frac{1}{2}mb^2\dot\theta^2 - mab\omega\dot\theta\sin(\theta-\omega t)$$
$$+ mga\sin\omega t - mgb\cos\theta$$
$$= \frac{1}{2}mb^2\dot\theta^2 - \frac{1}{2}ma^2\omega^2 + mga\sin\omega t - mgb\cos\theta$$

$$\dot\theta = \frac{p}{mb^2} - \frac{a}{b}\omega\sin(\theta-\omega t)$$

$$\dot\theta^2 = \frac{p^2}{m^2b^4} + \left(\frac{a}{b}\right)^2\omega^2\sin^2(\theta-\omega t) - 2\frac{ab\omega p}{mb^4}\sin(\theta-\omega t)$$

$$\boxed{\begin{aligned} H(\theta,p_\theta,t) = {} & \frac{p_\theta^2}{2mb^2} + \frac{1}{2}ma^2\omega^2\sin^2(\theta-\omega t) - \frac{a}{b}\omega p_\theta\sin(\theta-\omega t) \\ & - mgb\cos\theta \\ & -\frac{1}{2}ma^2\omega^2 + mga\sin\omega t \end{aligned}}$$

3.4
$$\boxed{\begin{aligned} \dot p_\theta = -\frac{\partial H}{\partial \theta} = {} & -ma^2\omega^2\sin(\theta-\omega t)\cos(\theta-\omega t) + \frac{a}{b}\omega p_\theta\cos(\theta-\omega t) \\ & - mgb\sin\theta \end{aligned}}$$

$$\boxed{\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mb^2} - \frac{a}{b}\omega\sin(\theta-\omega t)}$$